이학박사 학위논문

On the asymptotic dynamics of particle and kinetic Kuramoto synchronization models

(쿠라모토 입자, 운동 동기화 모델의 점근적 동역학에 관하여)

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서울대학교 대학원

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이 논문을 이학박사 학위논문으로 제출함

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On the asymptotic dynamics of particle and kinetic Kuramoto synchronization models

A dissertation
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Abstract

In this thesis, we study the Kuramoto model which describes the synchronous phenomena. In the Kuramoto model, the dynamics of the oscillators are presented by the intrinsic constant dynamics and the couplings between the oscillators. We study the sufficient conditions to achieve the emergence of synchronization for the Kuramoto model in various circumstances; network structure, frustrations, heterogeneous intrinsic dynamics, inertia effects, etc. We also study the dynamics of the kinetic Kuramoto-Sakaguchi equation, which is the macroscopic description for the mean field limit of the Kuramoto model.

Key words: Kuramoto model, kinetic equation, Kuramoto-Sakaguchi equation, synchronization, dynamical system
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Chapter 1

Introduction

Synchronization is a kind of the collective phenomena in which the members of a group demonstrate periodic motions with a common frequency by a coupled system. Synchronous behaviors are often observed in various field such as biological systems, physics, chemistry, engineering, and social sciences, for example, simultaneous flashing of fireflies, rhythmical contraction of pacemaker cells, linked pendulums, power network system, and applause in the concert, etc. These synchronized phenomena have received attentions and studied in various diciplines [2, 13, 61, 68]. However, the history of mathematical approach on synchronization is not long. The pioneering works proposed by Winfree [77] and Kuramoto [48, 49] lead the systematic studies on synchronization. They depicted the periodic motions as the dynamics of phase on the unit circle and introduced first order systems of ODEs to interpret synchronization of weakly coupled oscillators. By using polar coordinate, let θ_i be the position of an oscillator on the unit circle and consider the dynamics of the oscillator as

$$\dot{\theta}_i = \Omega_i + \hat{\omega}_i, \quad i = 1, \cdots, N,$$

where Ω_i is the natural frequency of i-th oscillator and $\hat{\omega}_i$ is the perturbation driven by couplings between oscillators. In this thesis, we focus on the phase synchronization model proposed by Kuramoto:

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad i = 1, \cdots, N, \tag{1.0.1}$$

where K is the uniform positive coupling strength. We assume that the natural frequency Ω_i is choosen from some distribution function $g = g(\Omega)$. In Kuramoto model, the couplings between oscillators are given by the sine value of phase differnce. Note that the right hand side of (1.0.1) is Lipschitz continuous and uniformly bounded. Thus, the well-posedness of (1.0.1) is guaranteed by Cauchy-Lipschitz theory.

The rest of this thesis consists of ten chapters. In Chapter 2, we briefly explain the mathematical definition of synchronization and present the derivation of the Kuramoto model. We also introduce the kinetic Kuramoto model, which describes the mean field limit for the Kuramoto model. We review some previous literature related to the synchronization of the Kuramoto system. In Chapter 3, we study the complete frequency synchronization with some relaxed contraints on the initial configurations. So far, most of previous analytic studies required the initial phases of the Kuramoto oscillators to be confined in the half circle. By devoting the dynamics of the order parameter, we show the synchronization with the larger class of the initial data. In Chapter 4, We consider the network structure and the interaction frustrations on the coupling between the Kuramoto oscillators. We study the synchronization with small perturbation of all-to-all network. We extend the previous research by considering non-uniform frustration and by relaxing the initial constraints. In Chapter 5, We present the concept of practical synchronization for the Kuramoto system with non-constant intrinsic dynamics. We study the sufficient conditions which guarantee the practical synchronization under heterogeneous external forcing. In Chapter 6, We extend the result of Chapter 5 by employing inertia into the dynamics. We study the dynamic interplay between inertia and heterogeneous forcing in the Kuramoto system. In Chapter 7, we provide the dynamics on the coupling strength so that the magnitude of the coupling depends on the phase difference. With this adaptive coupling, we find out the sufficient conditions to lead the synchronization and study the convergence rate. In Chapter 8, we study the kinetic Kuramoto model, so called Kuramoto-Sakaguchi equation. We provide a global BV-solution to the kinetic Kuramoto system using the front-tracking method. We show the exponential growing of approximate solution. In Chapter 9, we present the emergence of synchronization for the Kuramoto-Sakaguchi equation by using the dynamics of order parameter for the kinetic Kuramoto model. Finally, Chapter 10 is devoted to the summary of the thesis and future directions.

Chapter 2

Preliminaries

In this chapter, we introduce a derivation of Kuramoto model and review previous results on the synchronization for Kuramoto oscillators.

2.1 Synchronization

First, we introduce several definitions for the type of synchronization which will be used throughout the thesis.

Definition 2.1.1. [16] *Let $\Theta(t) = (\theta_1(t), \cdots, \theta_N(t))$ be a solution to the Kuramoto model* (1.0.1)*.*

1. *The solution Θ exhibits asymptotic complete phase synchronization if and only if the relative phase differences go to zero asymptotically:*
$$\lim_{t\to\infty} |\theta_i(t) - \theta_j(t)| = 0 \quad \textit{for all} \quad i, j = 1, \cdots, N$$
2. *The solution Θ exhibits asymptotic complete frequency synchronization if and only if the relative frequency differences go to zero asymptotically:*
$$\lim_{t\to\infty} |\dot{\theta}_i(t) - \dot{\theta}_j(t)| = 0 \quad \textit{for all} \quad i, j = 1, \cdots, N$$
3. *The solution Θ is called a phase-locked state for the Kuramotmo model* (1.0.1) *if and only if the relative phase differences go to the constant as $t \to \infty$:*
$$\lim_{t\to\infty} |\theta_i(t) - \theta_j(t)| = \theta_{ij}^{\infty} \quad \textit{for all} \quad i, j = 1, \cdots, N$$

Note that the asymptotic complete phase synchronization is a special case of phase-locked states and the phase-locked state implies that the oscillators show the asymptotic frequency synchronization. However, the asymptotic frequency synchronization is not equivalent to the phase-locked state, because if the convergence speed is slow, like $\frac{1}{t}$, then the difference of phases does not converge to a constant.

2.2 Derivation of the Kuramoto model

We present a heuristic derivation of the Kuramoto model from the linearly coupled Stuart-Landau oscillators in [63].

Let $z \in \mathbb{C}$ be a the Stuart-Landau oscillator, which follows the following dynamics:

$$\dot{z} = (1 - |z|^2 + \mathrm{i}\Omega)z, \tag{2.2.1}$$

where $\Omega \in \mathbb{R}$ is the natural frequency of the Stuart-Landau oscillator. We employ the polar coordinate $z = re^{\mathrm{i}\theta}$ and plug it into (2.2.1). Note that (2.2.1) can be separated into the dynamics of modulus r and the phase θ as follows.

$$\dot{r} = r(1 - r^2), \qquad \dot{\theta} = \Omega.$$

Then, it is easy to see that Stuart-Landau oscillator has an unstable equilibrium $r = 0$ and a stable limit cycle $r = 1$, on which it rotates with its natural frequency Ω. We now consider a weakly coupled system of N Stuart-Landau oscillators with an all-to-all linear coupling:

$$\frac{dz_j}{dt} = (1 - |z_j|^2 + \mathrm{i}\Omega_j)z_j + \frac{K}{N}\sum_{i=1}^{N}(z_i - z_j), \quad j = 1, \cdots, N, \tag{2.2.2}$$

where K is the positive coupling strength. Assume that every Stuart-Landau oscillators are confined in the stable limit cycle for uncoupled Stuart-Landau oscillator $r_j = 1$, i.e., $z_j = e^{\mathrm{i}\theta_j}$ for $j = 1, \cdots, N$. By comparing imaginary parts on both sides of (2.2.2), we can induce the following equation.

$$\dot{\theta}_j = \Omega_j + \frac{K}{N}\sum_{i=1}^{N}\sin(\theta_i - \theta_j) \tag{2.2.3}$$

Note that the dynamics is governed by the intrinsic constants Ω_j and the nonlinear coupling in second term of R.H.S. of (2.2.3). If the coupling strength K is zero, then the oscillators move on the unit circle with their natural frequencies. For sufficiently large K, however, the nonlinear coupling dominate the natural frequencies so that (2.2.3) yields the synchronized states.

Since the dynamics of Kuramoto oscillators (2.2.3) are defined by the difference of phase, the Kuramoto system is invariant under the translation. In other word, if we set $\tilde{\theta}_j := \theta_j + c$ for all $j = 1, \cdots, N$, with some constant c, then we have

$$\dot{\tilde{\theta}}_j = \Omega_j + \frac{K}{N}\sum_{i=1}^{N} \sin(\tilde{\theta}_i - \tilde{\theta}_j).$$

2.3 Kinetic Kuramoto model

Consider the situation that the number of Kuramoto oscillators N goes to infinity. To describe the dynamics of this mean field limit, we will treat the evolvement of a one-particle distribution function $f = f(\theta, \Omega, t)$. By the BBGKY hierarchy argument [46], we can attain the following kinetic equation, so called Kuramoto-Sakaguchi equation:

$$\begin{aligned} &\partial_t f + \partial_\theta(\omega([f]f) = 0, \qquad (\theta, \Omega) \in \mathbb{T} \times \mathbb{R}, \quad t \in (0, \infty), \\ &\omega[f](\theta, \Omega, t) := \Omega - K \int_{\mathbb{T}} \int_{\mathbb{R}} \sin(\theta - \theta_*) f(\theta_*, \Omega_*, t) d\Omega_* d\theta_*. \end{aligned} \tag{2.3.4}$$

The derivation of (2.3.4) is presented in [50] by using Neunzert's method [60].

2.4 Review on the previous literatures

In this part, we briefly review the state-of-the-art in terms of the synchronization problem for the Kuramoto model. For a detailed discussion, we refer readers to survey papers and books [2, 7, 49, 68]. The frequency synchronization problem [2, 20] has been treated using different approaches. Ermentrout [27] found a critical coupling at which all oscillators become phase-locked, independent of their number. The existence of phase locked state and its linear stability have been studied in several papers [4, 5, 10, 22, 45, 56, 57, 58, 70, 71, 72, 73] using tools such as a Lyapunov functional, spectral graph theory, and control theory. The studies most closely related to this chapter are those of Chopra and Spong [19], Choi et al. [16], and Dörfler and Bullo [25]. These papers use the phase-diameter $D(\Theta) := \max_{1 \le i,j \le N} |\theta_i - \theta_j|$ as a Lyapunov functional, and study its temporal evolution via Gronwall's inequality. In fact, these papers only deal with initial configurations whose phase-diameter is less than π. To date, π is the best upper bound; if we could extend this upper bound to 2π, it would be possible to rigorously justify the independence of initial configurations observed in numerical simulations. Before we close this section, we recall the most recent result on complete frequency synchronization from [16]. Below, we set

$$D(\Omega) := \max_{1 \le i,j \le N} |\Omega_i - \Omega_j| \quad \text{and} \quad D(\dot{\Theta}(t)) := \max_{1 \le i,j \le N} |\dot{\theta}_i - \dot{\theta}_j|.$$

Theorem 2.4.1. [31] *Let* $\Theta(t) = (\theta_1(t), \cdots, \theta_N(t))$ *be the global smooth solution to* (1.0.1)*. Then, the following estimates hold:*

1. (identical oscillators): *Suppose*

$$D(\Omega) = 0, \quad K > 0, \quad D(\Theta_0) < \pi$$

Then, we have an asymptotic phase synchronization:

$$D(\Theta_0)\exp(-Kt) \le D(\Theta(t)) \le D(\Theta_0)\exp(-K\frac{\sin D(\Theta_0)}{D(\Theta_0)}t)$$

2. (nonidentical oscillators): *Suppose*

$$D(\Omega) > 0, \quad K > D(\Omega), \quad D(\Theta_0) < D_0^\infty$$

Then, we have an asymptotic frequency synchronization:

$$D(\dot{\Theta}(t)) \le D(\dot{\Theta}(0))\exp(-K\cos D_0^\infty t)$$

Theorem 2.4.2. [16] *Suppose that the coupling strength and initial configuration Θ_0 satisfy*

$$0 < D(\Theta_0) =: D_0 < \pi, \qquad K > \frac{D(\Omega)}{\sin D_0}.$$

Then, for any solution $\Theta = (\theta_1, \cdots, \theta_N)$ to (1.0.1) *with initial condition Θ_0, there exist positive constants C_0 and Λ such that*

$$D(\dot{\Theta}(t)) \leq C_0 \exp(-\Lambda t), \quad \text{as } t \to \infty.$$

Remark 2.4.1. *For identical oscillators $D(\Omega) = 0$, we only need a positive coupling strength $K > 0$. In fact, complete frequency synchronization has been shown in [24] for an arbitrary initial configuration with $D(\Theta_0) < 2\pi$. Of course, the synchronization estimate given in [24] does not yield the detailed relaxation process toward a phase-locked state.*

Chapter 3

Complete synchronization

In this chapter, we present an improvement on the estimates for expoential frequency synchronization by exploiting the dynamics of the Kuramoto order paremeters. It was known by numerical simulations that the Kuramoto oscillators show the phase-locked state for sufficiently large K independent of the initial data. However, it was not proved analytically. So far, complete frequency synchronization for identical oscillators, which have a common natural frequency, has been demonstrated in [24], whereas, for nonidentical oscillators, the exponential relaxation toward the synchronization has been studied with restrictive initial configurations such that the oscillators are initially confined in a half circle [16]. First, we address an exponential complete phase synchronization for identical Kuramoto model for some larger class of initial configurations containing a half circle. We show that the diameter of oscillators shrink into a half circle in finite time so that we can apply the result in [16]. Second, we break the barrier on the diameter of admissible initial configuration π, which lead the complete frequency synchronization for the non-identical oscillators. This chapter is based on the joint work in [36]

3.1 Basic key estimates

In this section, we study the dynamics of the Kuramoto order parameters and phase-diameter under some a priori assumption. These estimates will be crucial to our complete frequency synchronization estimates in Sections 3.2.

3.1.1 Dynamics of order parameters

We introduce Kuramoto order parameters for the finite-dimensional Kuramoto model:

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i).$$

Recall that, for the phase configuration $\Theta = (\theta_1, \cdots, \theta_N)$, the Kuramoto order parameters r and ϕ are defined by the following relation:

$$re^{\mathrm{i}\phi} := \frac{1}{N}\sum_{j=1}^{N} e^{\mathrm{i}\theta_j}. \tag{3.1.1}$$

Since $re^{\mathrm{i}\phi}$ is the barycenter of the oscillators on the unit circle, r is always bounded, i.e., $0 \le r \le 1$. For each $i = 1, \cdots, N$, we multiply by $e^{-\mathrm{i}\theta_i}$ on both sides of (3.1.1). Then, by comparing the imaginary parts of both sides, we attain the following relations:

$$r\sin(\phi - \theta_i) = \frac{1}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i). \tag{3.1.2}$$

Using (3.1.2), we can express the Kuramoto model with the following form:

$$\dot{\theta}_i = \Omega_i + Kr\sin(\phi - \theta_i) \quad \text{for} \quad i = 1, \cdots, N. \tag{3.1.3}$$

From the argument in [44], we devide (3.1.1) by $e^{\mathrm{i}\phi}$ on both sides and compare real and imaginary parts of both sides to attain the following relations:

$$\frac{1}{N}\sum_{j=1}^{N} \cos(\theta_j - \phi) = r, \qquad \frac{1}{N}\sum_{j=1}^{N} \sin(\theta_j - \phi) = 0. \tag{3.1.4}$$

Moreover, by taking time derivative on (3.1.1), we have the evolutionary system:

$$\begin{aligned} \dot{r} &= -\frac{1}{N}\sum_{j=1}^{N} \sin(\theta_j - \phi)\Big(\Omega_j - Kr\sin(\theta_j - \phi)\Big), \\ \dot{\phi} &= \frac{1}{rN}\sum_{j=1}^{N} \cos(\theta_j - \phi)\Big(\Omega_j - Kr\sin(\theta_j - \phi)\Big). \end{aligned} \tag{3.1.5}$$

Note that for identical oscillators with $\Omega_j = 0$, it follows from (3.1.5) that

$$\begin{aligned} \dot{r} &= \frac{Kr}{N}\sum_{j=1}^{N}\sin^2(\theta_j - \phi), \quad t > 0, \\ \dot{\phi} &= -\frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \phi)\cos(\theta_j - \phi). \end{aligned} \tag{3.1.6}$$

The monotonicity of r can be easily seen from the first equation of (3.1.6). Note that the order parameter r is non-decreasing, but may not be strictly increasing: for example, let Θ_0 be the initial configuration such that $m(\neq \frac{N}{2})$ identical oscillators are located at 0 and $N - m$ are located at π. Then, it is easy to see that this configuration is an equilibrium for (1.0.1) and

$$z_c = \frac{me^{\mathrm{i}0} + (N-m)e^{\mathrm{i}\pi}}{N} = \frac{2m - N}{N} \neq 0, \quad r = \Big|\frac{2m-N}{N}\Big| > 0.$$

Thus, we have

$$r(t) = r(0), \qquad \forall\ t > 0.$$

In the following, we present the dynamics of r for nonidentical oscillators. For positive constants $\alpha, \delta < \frac{1}{2}$, we set $\beta_\delta := (1-\delta)\pi > \frac{\pi}{2}$ and define r_* and r^* such that

$$r_* := \frac{\max_j |\Omega_j|}{\sqrt{\alpha}K\sin\beta_\delta}, \qquad r^* := 1 - \alpha(2 + \sin^2\beta_\delta) > 0.$$

For a given configuration $\Theta = (\theta_1, \cdots, \theta_N) \in (\phi - \pi, \phi + \pi]^N$, we set extremal indices M and m:

$$M := \arg\max_{1\le i\le N}(\theta_i - \phi), \qquad m := \arg\min_{1\le i\le N}(\theta_i - \phi).$$

For such M and m, we define the *phase-diameter* $D(\Theta)$ as:

$$D(\Theta) := \theta_M - \theta_m.$$

Below, we denotes $r_0 := r(0)$.

Lemma 3.1.1. *Suppose that the initial configuration and parameters δ, α, and K satisfy*

$$(i)\ \max\{\theta_M(0) - \phi(0), \phi(0) - \theta_m(0)\} \leq \beta_\delta, \quad 0 < \delta < \frac{1}{2}.$$
$$(ii)\ 0 < \alpha < \frac{1}{2}, \quad 0 < r_* < r^* < 1.$$

Then the following assertions hold.

1. *If the order parameter r initially satisfies $r_* \leq r_0 \leq r^*$, then r is in non-decreasing mode at $t = 0$:*

$$\dot{r}(0) \geq 0.$$

2. *If $r_* < r_0$ and as long as*

$$\max_{0 \leq s \leq t} \max\{\theta_M(s) - \phi(s), \phi(s) - \theta_m(s)\} \leq \beta_\delta,$$

we have

$$\min_{0 \leq s \leq t} r(s) \geq \min\{r_0, r^*\}.$$

Proof. It suffices to show that, as long as $r_* \leq r_0 \leq r^*$, r is in non-decreasing mode at $t = 0$,

$$\dot{r}(0) \geq 0.$$

Once we have this, by the exactly same argument, we can show

$$\dot{r}(s) \geq 0 \qquad 0 \leq s \leq t, \tag{3.1.7}$$

if $r_* \leq r(s) \leq r^*$ is provided. From this fact, the second assertion can be proven as follows :

- Case A ($r_* < r_0 \leq r^*$) : Suppose there is a time $s_0 > 0$ such that

$$0 < s_0 \leq t, \quad r(s_0) < r_0 = \min\{r_0, r^*\}. \tag{3.1.8}$$

Define

$$s_1 := \sup_{0 \leq s \leq s_0} \{s : r(s) \geq r_0\}.$$

Since r is continuous function, it is clear that $r(s_1) = r_0$ and $s_1 < s_0$. Now, we choose s_2 satisfying $s_1 < s_2 < s_0$ and

$$r_* \leq \min_{s_1 \leq s \leq s_2} \{r(s)\} \leq \max_{s_1 \leq s \leq s_2} \{r(s)\} \leq r_0 \quad \text{and} \quad r_* < r(s_2) < r_0 = r(s_1).$$

This implies that there should be a time s_3 such that

$$s_1 \leq s_3 \leq s_2, \quad r_* \leq r(s_3) < r_0 \quad \text{and} \quad \dot{r}(s_3) < 0.$$

However, this contradicts to the property (3.1.7). So s_0 cannot satisfy (3.1.8), and hence we have

$$\min_{0 \leq s \leq t} r(s) \geq r_0 = \min\{r_0, r^*\}.$$

- Case B ($r^* < r_0$) : Again, we suppose there is a time s_0 satisfying

$$0 < s_0 \leq t, \quad r(s_0) < r^* = \min\{r_0, r^*\}.$$

Then, similar to the case A, there should be a time s_1 such that

$$0 < s_1 \leq s_0, \quad r_* \leq r(s_1) < r^* \quad \text{and} \quad \dot{r}(s_1) < 0,$$

which is again a contradiction to the fact (3.1.7). Hence, we have

$$\min_{0 \leq s \leq t} r(s) \geq r^* = \min\{r_0, r^*\}.$$

Now let us prove the first assertion. It follows from (3.1.5) and the Cauchy–Schwartz inequality that

$$\begin{aligned}
\dot{r} &= -\frac{1}{N}\sum_{j=1}^{N} \Omega_j \sin(\theta_j - \phi) + \frac{Kr}{N}\sum_{j=1}^{N} \sin^2(\theta_j - \phi) \\
&\geq -\frac{1}{N}\Big(\sum_{j=1}^{N} \Omega_j^2\Big)^{\frac{1}{2}} \Big(\sum_{j=1}^{N} \sin^2(\theta_j - \phi)\Big)^{\frac{1}{2}} + \frac{Kr}{N}\sum_{j=1}^{N} \sin^2(\theta_j - \phi) \\
&= \frac{K}{\sqrt{N}}\Big(\sum_{j=1}^{N} \sin^2(\theta_j - \phi)\Big)^{\frac{1}{2}} \Big[r\Big(\frac{1}{N}\sum_{j=1}^{N} \sin^2(\theta_j - \phi)\Big)^{\frac{1}{2}} - \frac{1}{K\sqrt{N}}\Big(\sum_{j=1}^{N} \Omega_j^2\Big)^{\frac{1}{2}}\Big]
\end{aligned}$$

$$\geq \frac{K}{\sqrt{N}}\Big(\sum_{j=1}^{N}\sin^2(\theta_j-\phi)\Big)^{\frac{1}{2}}\Big[r\Big(\frac{1}{N}\sum_{j=1}^{N}\sin^2(\theta_j-\phi)\Big)^{\frac{1}{2}}-\frac{\max_{1\leq j\leq N}|\Omega_j|}{K}\Big].$$

Here, we used the simple inequality $\Big(\sum_{j=1}^{N}\Omega_j^2\Big)^{\frac{1}{2}} \leq \sqrt{N}\max_j|\Omega_j|$.

Suppose that

$$r_* \leq r_0 \leq r^*. \tag{3.1.9}$$

We claim:

$$r_0\Big(\frac{1}{N}\sum_{j=1}^{N}\sin^2(\theta_j-\phi)\Big)^{\frac{1}{2}}\Big|_{t=0}-\frac{\max_{1\leq j\leq N}|\Omega_j|}{K}\geq 0. \tag{3.1.10}$$

Proof of (3.1.10): We separate oscillators into two groups I_+ and I_- such that

$$I_+ := \Big\{j : |\theta_{j0}-\phi_0| < \frac{\pi}{2}\Big\}, \qquad I_- := \Big\{j : \frac{\pi}{2} \leq |\theta_{j0}-\phi_0| \leq \beta_\delta\Big\}.$$

It follows from (3.1.4) and (3.1.9) that

$$\begin{aligned}\frac{1}{N}\sum_{j=1}^{N}\cos(\theta_{j0}-\phi_0)\leq r^* &\Longleftrightarrow \sum_{j\in I_+}\cos(\theta_{j0}-\phi_0)+\sum_{j\in I_-}\cos(\theta_{j0}-\phi_0)\leq Nr^*\\ &\Longleftrightarrow \sum_{j\in I_+}\cos(\theta_{j0}-\phi_0)\leq Nr^*-\sum_{j\in I_-}\cos(\theta_{j0}-\phi_0)\\ &\Longrightarrow \sum_{j\in I_+}\cos(\theta_{j0}-\phi_0)\leq Nr^*-|I_-|\cos\beta_\delta,\end{aligned} \tag{3.1.11}$$

where $\phi_0=\phi(0)$. We use (3.1.11) to obtain

$$\begin{aligned}\sum_{j=1}^{N}\cos^2(\theta_{j0}-\phi_0) &= \sum_{j\in I_+}\cos^2(\theta_{j0}-\phi_0)+\sum_{j\in I_-}\cos^2(\theta_{j0}-\phi_0)\\ &\leq \sum_{j\in I_+}\cos(\theta_{j0}-\phi_0)+\sum_{j\in I_-}\cos^2(\theta_{j0}-\phi_0)\\ &\leq Nr^*-|I_-|\cos\beta_\delta+|I_-|\cos^2\beta_\delta.\end{aligned}$$

On the other hand, we use the inequality $\cos\beta_\delta - \cos^2\beta_\delta \geq 2\cos\beta_\delta$ to derive

$$\begin{aligned}\sum_{j=1}^N \sin^2(\theta_{j0}-\phi_0) &= N - \sum_{j=1}^N \cos^2(\theta_{j0}-\phi_0)\\ &\geq N - Nr^* + |I_-|(\cos\beta_\delta - \cos^2\beta_\delta)\\ &\geq N - Nr^* + 2|I_-|\cos\beta_\delta.\end{aligned}$$

This yields

$$r_0\left(\frac{1}{N}\sum_{j=1}^N \sin^2(\theta_{j0}-\phi_0)\right)^{\frac{1}{2}} \geq r_0\left(\frac{N - Nr^* + 2|I_-|\cos\beta_\delta}{N}\right)^{\frac{1}{2}}. \tag{3.1.12}$$

• Case A ($|I_-| > \alpha N$): From our definition of r_*, we have

$$\begin{aligned}r_0\left(\frac{1}{N}\sum_{j=1}^N \sin^2(\theta_{j0}-\phi_0)\right)^{\frac{1}{2}} &\geq r_*\left(\frac{1}{N}\sum_{j\in I_-} \sin^2(\theta_{j0}-\phi_0)\right)^{\frac{1}{2}}\\ &\geq r_*\sqrt{\alpha}\sin\beta_\delta = \frac{\max_{1\leq j\leq N}|\Omega_j|}{K}.\end{aligned} \tag{3.1.13}$$

• Case B ($|I_-| \leq \alpha N$): We use definitions of r_*, r^* and (3.1.12) to obtain

$$\begin{aligned}r_0\left(\frac{1}{N}\sum_{j=1}^N \sin^2(\theta_{j0}-\phi_0)\right)^{\frac{1}{2}} &\geq r_*\left(\frac{N - Nr^* + 2|I_-|\cos\beta_\delta}{N}\right)^{\frac{1}{2}}\\ &\geq r_*(1-r^*-2\alpha)^{\frac{1}{2}} = \frac{\max_{1\leq j\leq N}|\Omega_j|}{\sqrt{\alpha}K\sin\beta_\delta}(\alpha\sin^2\beta_\delta)^{\frac{1}{2}}\\ &= \frac{\max_{1\leq j\leq N}|\Omega_j|}{K}.\end{aligned} \tag{3.1.14}$$

Then, it follows from (3.1.13) and (3.1.14) that we have the desired estimate (3.1.10). □

Remark 3.1.1. *1. Note that, from the proof, if $r_* < r_0 \leq r^*$, then*

$$r(t) \geq r_0, \quad t \geq 0.$$

2. By choosing $\alpha \approx 0$ and sufficiently large K, we can get $(r_*, r^*) \approx (0,1)$.

We next estimate the evolution of the overall phase ϕ in the following lemma.

Lemma 3.1.2. *Let ϕ be the overall phase of the configuration $\Theta = \Theta(t)$ whose dynamics is governed by* (1.0.1). *Then, we have*

$$|\dot{\phi}| \leq K(1-r) + \frac{1}{r} \max_{1\leq i\leq N} |\Omega_i|, \quad t > 0.$$

Proof. It follows from (3.1.5) that

$$\begin{aligned}
\dot{\phi} &= \frac{1}{rN} \sum_{j=1}^{N} \cos(\theta_j - \phi)\Big(\Omega_j - Kr\sin(\theta_j - \phi)\Big) \\
&= \frac{1}{rN} \sum_{j=1}^{N} \Omega_j \cos(\theta_j - \phi) - \frac{K}{N} \sum_{j=1}^{N} \cos(\theta_j - \phi)\sin(\theta_j - \phi) \\
&=: \mathcal{I}_1 + \mathcal{I}_2.
\end{aligned} \tag{3.1.15}$$

• (Estimate of $\mathcal{I}_1$): We use a rough bound

$$|\Omega_j \cos(\theta_j - \phi)| \leq \max_{1\leq j\leq N} |\Omega_j|$$

to obtain

$$|\mathcal{I}_1| \leq \frac{1}{r} \max_{1\leq j\leq N} |\Omega_j|. \tag{3.1.16}$$

• (Estimate of $\mathcal{I}_2$): Below, we provide the upper and lower bounds for $\mathcal{I}_2$.

• Case A (Upper bound): We use (3.1.4) to obtain

$$\begin{aligned}
\mathcal{I}_2 &= -\frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \phi)\cos(\theta_j - \phi) \\
&= -\frac{K}{N} \sum_{j=1}^{N} \Big[(\cos(\theta_j - \phi) - 1)(\sin(\theta_j - \phi) - 1) \\
&\qquad\qquad + \cos(\theta_j - \phi) + \sin(\theta_j - \phi) - 1\Big] \\
&= -\frac{K}{N} \sum_{j=1}^{N} \underbrace{(\cos(\theta_j - \phi) - 1)(\sin(\theta_j - \phi) - 1)}_{\geq 0} - \frac{K}{N} \sum_{j=1}^{N} \cos(\theta_j - \phi)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j-\phi)+K \\
&\le -\frac{K}{N}\sum_{j=1}^{N}\cos(\theta_j-\phi)-\frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j-\phi)+K \\
&= K(1-r).
\end{aligned} \tag{3.1.17}
$$

- Case B (Lower bound): Similar to Case A, we have

$$
\begin{aligned}
\mathcal{I}_2 &= -\frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j-\phi)\cos(\theta_j-\phi) \\
&= -\frac{K}{N}\sum_{j=1}^{N}\Big[(\cos(\theta_j-\phi)-1)(\sin(\theta_j-\phi)+1) \\
&\qquad\qquad -\cos(\theta_j-\phi)+\sin(\theta_j-\phi)+1\Big] \\
&= -\frac{K}{N}\sum_{j=1}^{N}\underbrace{(\cos(\theta_j-\phi)-1)(\sin(\theta_j-\phi)+1)}_{\le 0}+\frac{K}{N}\sum_{j=1}^{N}\cos(\theta_j-\phi) \\
&- \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j-\phi)-K \\
&\ge \frac{K}{N}\sum_{j=1}^{N}\cos(\theta_j-\phi)-\frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j-\phi)-K \\
&= -K(1-r).
\end{aligned} \tag{3.1.18}
$$

Finally, we combine (3.1.17) and (3.1.18) to obtain

$$
-K(1-r)\le \mathcal{I}_2 \le K(1-r). \tag{3.1.19}
$$

In (3.1.15), we combine (3.1.16) and (3.1.19) to obtain the desired estimate. □

3.1.2 Evolution of phase-diameter

In this subsection, we provide a decay estimate of the phase-diameter $D(\Theta)$ under a priori condition of fluctuations.

We first remind the Gronwall's inequality whose proof can be found in various places, for example in the Appendix B of [28].

Lemma 3.1.3 (Gronwall's inequality)**.** *Let* $f : [0,T] \mapsto (-\infty,\infty)$ *be a nonnegative, differentiable function and let* $g, h : [0,T] \to (-\infty,\infty)$ *are summable functions.*
(i) If we have the differential inequality

$$f'(t) \le g(t)f(t) + h(t), \quad \forall t \in [0,T],$$

then the following inequality holds

$$f(t) \le f(0)e^{\int_0^t g(s)ds} + \int_0^t h(s)e^{\int_s^t g(\tau)d\tau}ds.$$

(ii) Likewise, if we have

$$f'(t) \ge g(t)f(t) + h(t), \quad \forall t \in [0,T],$$

then the following inequality is true

$$f(t) \ge f(0)e^{\int_0^t g(s)ds} + \int_0^t h(s)e^{\int_s^t g(\tau)d\tau}ds.$$

Lemma 3.1.4. *For a positive constant* $T \in (0,\infty]$*, let* $\Theta = (\theta_1, \cdots, \theta_N)$ *be a solution to* (1.0.1) *satisfying the a priori condition:*

$$\beta_T := \max_{0 \le \tau \le T} \max\{\theta_M(\tau) - \phi(\tau), \phi(\tau) - \theta_m(\tau)\} < \pi. \tag{3.1.20}$$

Then, the phase-diameter $D(\Theta)$ *satisfies the following lower and upper bounds: For any* $0 < t < T$*,*

$$(i)\ D(\Theta_0)e^{-K\int_0^t r(s)ds} - D(\Omega)\int_0^t e^{-K\int_s^t r(\tau)d\tau}ds \le D(\Theta(t)).$$
$$(ii)\ D(\Theta(t)) \le D(\Theta_0)e^{-K\frac{\sin\beta_T}{\beta_T}\int_0^t r(s)ds} + D(\Omega)\int_0^t e^{-K\frac{\sin\beta_T}{\beta_T}\int_s^t r(\tau)d\tau}ds.$$

Proof. (i) (Lower bound estimate): We use equation (3.1.3) to derive

$$\begin{aligned}\dot{D}(\Theta) &= \dot{\theta}_M - \dot{\theta}_m \\ &= \Omega_M - \Omega_m - Kr\left(\sin(\theta_M - \phi) - \sin(\theta_m - \phi)\right) \\ &\ge \Omega_M - \Omega_m - Kr(\theta_M - \theta_m) \\ &\ge -D(\Omega) - KrD(\Theta),\end{aligned} \tag{3.1.21}$$

where the first inequality comes from the fact that

$$\sin x \begin{cases} \le x, & if \quad x \ge 0, \\ \ge x, & if \quad x \le 0. \end{cases}$$

Then, Gronwall's lemma for (3.1.21) yields the desired lower bound estimate for $D(\theta)$.

(ii) (Upper bound estimate): We first note that, under the a priori condition (3.1.20), i.e.,

$$-\beta_T \le \theta_m - \phi \le 0 \le \theta_M - \phi \le \beta_T, \quad \text{for some } 0 < \beta_T < \pi,$$

we have

$$\begin{aligned} \sin(\theta_M - \phi) - \sin(\theta_m - \phi) &\ge \frac{\sin \beta_T}{\beta_T}(\theta_M - \phi) - \frac{\sin \beta_T}{\beta_T}(\theta_m - \phi) \\ &= \frac{\sin \beta_T}{\beta_T}(\theta_M - \theta_m). \end{aligned} \tag{3.1.22}$$

Then, we use (3.1.22) to obtain

$$\begin{aligned} \dot{D}(\Theta) &= \Omega_M - \Omega_m - Kr\left(\sin(\theta_M - \phi) - \sin(\theta_m - \phi)\right) \\ &\le \Omega_M - \Omega_m - Kr\frac{\sin \beta_T}{\beta_T}(\theta_M - \theta_m) \\ &= \Omega_M - \Omega_m - Kr\frac{\sin \beta_T}{\beta_T}D(\Theta) \\ &\le D(\Omega) - Kr\frac{\sin \beta_T}{\beta_T}D(\Theta). \end{aligned} \tag{3.1.23}$$

Now, (3.1.23) and Gronwall's lemma imply the desired estimate. □

3.2 Complete frequency synchronization

In this section, we extend the frequency synchronization estimate to initial configurations whose diameter is larger than π.

3.2.1 Identical oscillators

Consider Kuramoto oscillators with

$$\Omega_i = 0, \quad 1 \le i \le N.$$

Although the results in [24, 45, 59, 64, 66] establish complete frequency synchronization and complete phase synchronization for an arbitrary initial configuration and almost all initial configuration, respectively, we do not have detailed information about the relaxation process and structure of phase-locked states. Of course, it is known that the only stable phase-locked state corresponds to complete phase synchronization consisting of a single phase. In the following, we study the detailed relaxation process by investigating the dynamics of the order parameters instead of the phase-diameter.

Lemma 3.2.1. *Let* $\Theta = (\theta_1, \cdots, \theta_N)$ *be a solution to* (1.0.1) *with initial data* Θ_0 *satisfying* $r_0 > 0$*. Then, we have*

$$\lim_{t\to\infty} \sin(\theta_i(t) - \phi(t)) = 0, \quad i = 1, \cdots, N.$$

Proof. It follows from $(3.1.6)_1$ that we have

$$r(t) = r_0 \exp\Big(\int_0^t g(\tau)d\tau\Big), \qquad g(\tau) := \frac{K}{N}\sum_{j=1}^N \sin^2(\phi(\tau) - \theta_j(\tau)).$$

Because r is bounded from above by 1, the nonnegative function g belongs to $L^1(0,\infty)$, i.e.,

$$\int_0^\infty g(t)dt = \frac{K}{N}\sum_{j=1}^N \int_0^\infty \sin^2(\phi(t) - \theta_j(t))dt < \infty. \tag{3.2.1}$$

We claim:

$$\lim_{t\to\infty} \sin^2(\theta_j(t) - \phi(t)) = 0, \qquad j = 1, \cdots, N.$$

Proof of claim: Suppose not, i.e., there exist $j, \nu_0 > 0$ and an increasing sequence of times $\{t_n\}_{n=1}^\infty$ such that

$$\sin^2(\phi(t_n) - \theta_j(t_n)) > \nu_0 \qquad \text{and} \qquad \lim_{n\to\infty} t_n = \infty.$$

On the other hand, note that

$$\begin{aligned}\frac{d}{dt}\sin^2(\phi(t)-\theta_j(t)) &= 2\sin(\phi(t)-\theta_j(t))\cos(\phi(t)-\theta_j(t))(\dot{\phi}(t)-\dot{\theta}_j(t))\\ &= (\dot{\phi}(t)-\dot{\theta}_j(t))\sin(2(\phi(t)-\theta_j(t))).\end{aligned}$$

We use the above relation and Lemma 3.1.2 to obtain

$$\left|\frac{d}{dt}\sin^2(\phi(t)-\theta_j(t))\right| \le (|\dot{\phi}(t)|+|\dot{\theta}_j|) \le K - Kr + Kr = K. \tag{3.2.2}$$

We use the bound on the derivative in (3.2.2) to derive

$$\begin{aligned}&\sin^2(\phi(t_n)-\theta_j(t_n)) > \nu_0 \implies \sin^2(\phi(t)-\theta_j(t)) \ge \frac{\nu_0}{2},\\ &t \in \left[t_n - \frac{\nu_0}{2K}, t_n + \frac{\nu_0}{2K}\right].\end{aligned} \tag{3.2.3}$$

Possibly extracting a subsequence(we abuse the notation), we may assume

$$t_{n+1} - t_n > \frac{\nu_0}{K} \qquad \forall n = 1, 2 \dots. \tag{3.2.4}$$

Combining (3.2.1), (3.2.3) and (3.2.4), we derive a contradiction:

$$\int_0^\infty \sin^2(\phi(t)-\theta_j(t))dt \ge \sum_{n=1}^\infty \int_{t_n-\nu_0/2K}^{t_n+\nu_0/2K} \sin^2(\phi(t)-\theta_j(t))dt \ge \sum_{n=1}^\infty \frac{\nu_0^2}{2K} = \infty.$$

□

In the following corollary, for any initial configuration, we show the complete frequency synchronization, and characterize the phase-locked states for identical Kuramoto oscillators.

Corollary 3.2.1. *Let $\Theta = (\theta_1, \cdots, \theta_N)$ be a solution to* (1.0.1) *with initial data Θ_0. Then, we have the following assertions:*

1. *If $r_0 > 0$, then we have a dichotomy:*

$$\lim_{t\to\infty} |\theta_j - \phi| = 0 \quad \textit{or} \quad \pi, \quad \textit{for all } j = 1, \cdots, N.$$

2. *If $r_0 = 0$, then the initial configuration Θ_0 is the equilibrium solution to* (1.0.1) *with $\Omega_i = 0, \quad 1 \le i \le N$.*

Proof. The first assertion follows from Lemma 3.2.1. Now we focus on the second assertion. Suppose that initial data Θ_0 and natural frequencies Ω_j satisfy

$$r_0 = 0, \quad \Omega_j = 0, \quad 1 \le j \le N.$$

We claim:

$$\Theta(t) := \Theta_0, \qquad \forall t > 0,$$

is the equilibrium solution of (1.0.1). First of all, it is clear that

$$\dot{\theta}_j(t) = 0, \qquad 1 \le j \le N, \quad \forall t > 0. \tag{3.2.5}$$

On the other hand, for all $t > 0$, we have

$$\begin{aligned} \Omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j(t) - \theta_i(t)) &= \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j(0) - \theta_i(0)) \\ &= K r_0 \sin(\phi(0) - \theta_j(0)), \end{aligned} \tag{3.2.6}$$

where the second equality follows from (3.1.3). Since $r_0 = 0$, Equation (3.2.6) implies

$$\Omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j(t) - \theta_i(t)) = 0, \quad \forall t > 0. \tag{3.2.7}$$

Combining (3.2.5) and (3.2.7), we verified $\Theta(t) := \Theta_0$ is the solution to (1.0.1) with Ω_i, $1 \le i \le N$. □

We next present admissible initial configurations that relax to the complete phase configuration exponentially fast. For the relaxation estimate, we split our analysis into two steps. First, we show that, as long as the extremal fluctuations $\theta_M - \phi$ and $\phi - \theta_m$ are less than π, the phase-diameter decays exponentially fast to zero. Second, we identify a class of initial phases whose evolution guarantees the a priori condition.

We are now ready to provide an exponential frequency synchronization for some initial configurations whose diameters are larger than π. Our strategy is as follows. We first show that the initial configuration evolves to a configuration whose diameter is less than π in finite time (entrance time), and then, we apply Theorem 2.4.2 with this intermediate configuration as new initial data at this entrance time. Below, we set

$$\varepsilon_1(r_0,\delta) := \frac{\beta_\delta - \frac{\pi}{2}}{1 + \frac{2(1-r_0)}{\pi r_0} \cdot \frac{\beta_\delta}{\sin \beta_\delta}}.$$

Lemma 3.2.2. *Suppose that, for $0 < \delta < \frac{1}{2}$, the initial configuration satisfies*

$$r_0 > 0, \qquad \max_{1 \le i \le N} |\theta_{i0} - \phi_0| < \frac{\pi}{2} + \varepsilon_1(r_0,\delta). \tag{3.2.8}$$

Then, for any solution $\Theta = (\theta_1, \cdots, \theta_N)$ of (1.0.1), there exists $t_e > 0$ such that

$$(\theta_M - \theta_m)(t_e) < \pi.$$

More precisely, we can choose t_e as follows:

$$t_e = \Big(\frac{2\beta_\delta}{\pi K r_0 \sin \beta_\delta}\Big) \cdot \Big(\frac{\beta_\delta - \frac{\pi}{2}}{1 + \frac{2(1-r_0)\beta_\delta}{\pi r_0 \sin \beta_\delta}}\Big).$$

Proof. We use a bootstrapping argument as follows. We set

$$\mathcal{T} := \{t \in [0,\infty) : \max_{0 \le \tau \le t} \max\{(\theta_M - \phi)(\tau), (\phi - \theta_m)(\tau)\} < \pi\}, \quad T^* := \sup \mathcal{T}.$$

Then, it follows from (3.2.8) that:

$$\max\{(\theta_M - \phi)(0), (\phi - \theta_m)(0)\} < \frac{\pi}{2} + \varepsilon_1,$$

and from the continuities of $\theta_M - \phi$ and $\phi - \theta_m$ that there exists $t' > 0$ such that

$$\max_{0 \le \tau \le t'} \max\{(\theta_M - \phi)(\tau), (\phi - \theta_m)(\tau)\} < \pi, \quad t' \in \mathcal{T}.$$

• (Rough estimate for $D(\Theta)$): We use the lower bound of $r(t) \ge r_0, \ t \le t'$, Lemma 3.1.2, and

$$\dot\theta_M = -Kr\sin(\theta_M - \phi) \quad \text{and} \quad \dot\theta_m = Kr\sin(\phi - \theta_m)$$

to derive

$$\begin{aligned}
&\frac{d}{dt}(\theta_M - \phi) \le -Kr\sin(\theta_M - \phi) + K(1-r) \le K(1-r) \le K(1-r_0),\\
&\frac{d}{dt}(\phi - \theta_m) \le -Kr\sin(\phi - \theta_m) + K(1-r) \le K(1-r) \le K(1-r_0).
\end{aligned} \tag{3.2.9}$$

Then, thanks to (3.2.9), we will choose $t_e \in \mathcal{T}$ and ε_1 such that, for $t \in (0, t_e)$, we have

$$\begin{aligned}\max_{0\le t\le t_e} \max\{\theta_M - \phi, \phi - \theta_m\} &\le \max\{(\theta_M - \phi)(0), (\phi - \theta_m)(0)\} + K(1-r_0)t_e \\ &\le \frac{\pi}{2} + \varepsilon_1 + K(1-r_0)t_e \le \beta_\delta < \pi.\end{aligned} \tag{3.2.10}$$

• (Refined estimate for $D(\Theta)$): Under the rough estimate (3.2.10), which satisfies the a priori assumption (3.1.20), we can apply Lemma 3.1.3 and $r \ge r_0$ to obtain

$$\begin{aligned}(\theta_M - \theta_m)(t_e) &\le (\theta_M - \theta_m)(0)e^{-K\frac{\sin\beta_\delta}{\beta_\delta}\int_0^{t_e} r(\tau)d\tau} \\ &\le (\theta_M - \theta_m)(0)e^{-K\frac{\sin\beta_\delta}{\beta_\delta}r_0 t_e} \\ &\le (\pi + 2\varepsilon_1)e^{-K\frac{\sin\beta_\delta}{\beta_\delta}r_0 t_e}.\end{aligned} \tag{3.2.11}$$

In (3.2.10) and (3.2.11), we need to choose ε_1 and t_e to satisfy

$$\frac{\pi}{2} + \varepsilon_1 + K(1-r_0)t_e \le \beta_\delta \quad \text{and} \quad (\pi + 2\varepsilon_1)e^{-K\frac{\sin\beta_\delta}{\beta_\delta}r_0 t_e} < \pi. \tag{3.2.12}$$

We next determine explicit functional forms for t_e and ε_1 satisfying relations (3.2.12). For definiteness, we will look for t_e and ε_1 satisfying the coupled relations:

$$\begin{aligned}&\frac{\pi}{2} + \varepsilon_1 + K(1-r_0)t_e = (1-\delta)\pi = \beta_\delta \quad \text{and} \\ &\pi + 2\varepsilon_1 = \pi\Big(1 + K\frac{\sin\beta_\delta}{\beta_\delta}r_0 t_e\Big).\end{aligned} \tag{3.2.13}$$

Because $1 + x < e^x, \quad x > 0$, once we find t_e and ε_1 satisfying (3.2.13), the pair (t_e, ε_1) also satisfies relation (3.2.12). From the second equation of (3.2.13), we have

$$t_e = \frac{2\beta_\delta}{\pi K r_0 \sin\beta_\delta}\varepsilon_1,$$

and we substitute this relation in the first equation of (3.2.13) to find

$$\varepsilon_1 = \frac{\beta_\delta - \frac{\pi}{2}}{1 + \frac{2(1-r_0)\beta_\delta}{\pi r_0 \sin\beta_\delta}} \quad \text{and} \quad t_e = \Big(\frac{2\beta_\delta}{\pi K r_0 \sin\beta_\delta}\Big)\cdot\Big(\frac{\beta_\delta - \frac{\pi}{2}}{1 + \frac{2(1-r_0)\beta_\delta}{\pi r_0 \sin\beta_\delta}}\Big).$$

□

Theorem 3.2.1. *Suppose that the initial configuration and $0 < \delta < \frac{1}{2}$ satisfy*

$$r_0 > 0, \qquad \max_{1 \le i \le N} |\theta_{i0} - \phi_0| < \frac{\pi}{2} + \frac{\beta_\delta - \frac{\pi}{2}}{1 + \frac{2(1-r_0)}{\pi r_0} \cdot \frac{\beta_\delta}{\sin \beta_\delta}},$$

and let $\Theta = (\theta_1, \cdots, \theta_N)$ be a solution to (1.0.1) with initial data Θ_0. Then, there exists positive constants C and Λ such that

$$D(\Theta(t)) \le Ce^{-\Lambda t}, \quad \text{as } t \to \infty.$$

Proof. Let Θ be the solution to (1.0.1) with initial data satisfying conditions (3.2.8). Then, it follows from Lemma 3.2.2 that there exists a finite time $t_e > 0$ such that

$$D(\Theta(t_e)) < \pi.$$

Thus, we can apply Theorem 2.4.2 with initial data $\Theta(t_e)$ after $t > t_e$ to derive the desired exponential frequency synchronization. □

Remark 3.2.1. *The complete phase synchronization estimates have been extensively studied in literature [45, 59, 64, 66] of control theory based on the gradient flow structure of the Kuramoto model and LaSalle's invariance principle, which establishes the complete phase synchronization for almost all initial configuration. However, this analysis does not yield the information on the basin of phase synchronization (see Theorem 5.1 in [25]). In contrast, the result in Theorem 3.2.1 describes the proper subset of basin of phase synchronization.*

3.2.2 Nonidentical oscillators

In this part, we study complete frequency synchronization for nonidentical oscillators by analyzing the dynamics of the order parameters r and ϕ introduced in Section 3.1. We set

$$\varepsilon_2(r_0, \delta, K, \{\Omega_i\}) := \frac{\beta_\delta - \frac{\pi}{2}}{1 + \frac{K(1-r_0) + \left(1 + \frac{1}{r_0}\right) \max_j |\Omega_j|}{\frac{\pi K r_0 \sin \beta_\delta}{2\beta_\delta} - \frac{D(\Omega)}{2}}}.$$

Lemma 3.2.3. *Suppose that the initial configuration* Θ_0 *and coupling strength* K *satisfy*

$$
\begin{aligned}
&(i)\ r_* < r_0 \le r^*, \qquad \max_{1\le i\le N} |\theta_{i0} - \phi_0| < \frac{\pi}{2} + \varepsilon_2,\\
&(ii)\ K > \max\left\{ \frac{\max_j |\Omega_j|}{[1-\alpha(2+\sin^2\beta_\delta)][\sqrt{\alpha}\sin\beta_\delta]}, \frac{\beta_\delta D(\Omega)}{\pi r_0 \sin\beta_\delta} \right\}.
\end{aligned}
$$

Then, there exists a finite time $t_e \in (0,\infty)$ *such that*

$$
(\theta_M - \theta_m)(t_e) < \pi.
$$

Proof. We set

$$
\mathcal{T} := \{t\ \in [0,\infty)\ :\ \max_{0\le\tau\le t} \max\{(\theta_M-\phi)(\tau), (\phi-\theta_m)(\tau)\} < \pi\}, \quad T^* := \sup\mathcal{T}.
$$

Then, it follows from the initial condition:

$$
\max\{(\theta_M - \phi)(0), (\phi - \theta_m)(0)\} < \frac{\pi}{2} + \varepsilon_2
$$

and the continuities of $\theta_M - \phi$ and $\phi - \theta_m$ that there exists $t' > 0$ such that

$$
\max_{0\le\tau\le t'} \max\{(\theta_M - \phi)(\tau), (\phi - \theta_m)(\tau)\} < \beta_\delta, \quad t' \in \mathcal{T}.
$$

• Step A: By the assumption on the initial configuration Θ_0, we have

$$
\max\{(\theta_M - \phi)(0), (\phi - \theta_m)(0)\} \le \frac{\pi}{2} + \varepsilon_2.
$$

On the other hand, we use

$$
\frac{d\theta_M}{dt} = \Omega_M - Kr\sin(\theta_M - \phi), \qquad \frac{d\theta_m}{dt} = \Omega_m - Kr\sin(\theta_m - \phi)
$$

and Lemma 3.1.2 to obtain

$$
\begin{aligned}
\frac{d}{dt}(\theta_M - \phi) &\le \Omega_M - Kr\sin(\theta_M - \phi) + K(1-r) + \frac{\max_j |\Omega_j|}{r}\\
&\le K(1-r) + \left(1 + \frac{1}{r}\right)\max_j |\Omega_j|, \qquad (3.2.14)\\
\frac{d}{dt}(\phi - \theta_m) &\le K(1-r) + \left(1 + \frac{1}{r}\right)\max_j |\Omega_j|.
\end{aligned}
$$

Because $r_* < r_0 \le r^*$, it follows from Lemma 3.1.1 that

$$r(t) \ge r_0, \quad 0 \le t \le t'. \tag{3.2.15}$$

Then, (3.2.14) and (3.2.15) imply that, for $0 < h < t'$,

$$\max_{0\le t\le h}\{\theta_M - \phi, \phi - \theta_m\} \le \frac{\pi}{2} + \varepsilon_2 + \left[K(1-r_0) + \left(1 + \frac{1}{r_0}\right)\max_j |\Omega_j|\right] h. \tag{3.2.16}$$

As long as

$$\frac{\pi}{2} + \varepsilon_2 + \left(K(1-r_0) + \left(1 + \frac{1}{r_0}\right)\max_j |\Omega_j|\right) h \le \beta_\delta < \pi, \tag{3.2.17}$$

we can also use (3.1.23) to obtain

$$\begin{aligned}\frac{dD(\Theta)}{dt} &= \dot\theta_M - \dot\theta_m \\ &= \Omega_M - \Omega_m - Kr\left(\sin(\theta_M - \phi) - \sin(\theta_m - \phi)\right) \\ &\le D(\Omega) - Kr\frac{\sin\beta_\delta}{\beta_\delta}D(\Theta).\end{aligned}$$

This yields

$$\begin{aligned}D(\Theta(h)) &\le D(\Theta_0)e^{-K\frac{\sin\beta_\delta}{\beta_\delta}\int_0^h r(s)ds} + D(\Omega)\int_0^h e^{-K\frac{\sin\beta_\delta}{\beta_\delta}\int_s^h r(\tau)d\tau}ds \\ &\le D(\Theta_0)e^{-K\frac{\sin\beta_\delta}{\beta_\delta}r_0 h} + D(\Omega)\int_0^h e^{-K\frac{\sin\beta_\delta}{\beta_\delta}r_0(h-s)}ds \\ &\le \left[D(\Theta_0) + \frac{D(\Omega)\beta_\delta}{Kr_0\sin\beta_\delta}\left(e^{K\frac{\sin\beta_\delta}{\beta_\delta}r_0 h} - 1\right)\right]e^{-K\frac{\sin\beta_\delta}{\beta_\delta}r_0 h}.\end{aligned} \tag{3.2.18}$$

• Step B (Determination of t_e and ε_2): It follows from (3.2.16), (3.2.17), and (3.2.18) that, if we can determine t_e and ε_2 to satisfy

$$\begin{aligned}&\frac{\pi}{2} + \varepsilon_2 + \left(K(1-r_0) + \left(1 + \frac{1}{r_0}\right)\max_j |\Omega_j|\right) t_e \le \beta_\delta, \\ &D(\Theta(t_e)) \le \left[D(\Theta_0) + \frac{D(\Omega)\beta_\delta}{Kr_0\sin\beta_\delta}\left(e^{K\frac{\sin\beta_\delta}{\beta_\delta}r_0 t_e} - 1\right)\right]e^{-K\frac{\sin\beta_\delta}{\beta_\delta}r_0 t_e} < \pi,\end{aligned} \tag{3.2.19}$$

then we are done. For this, we claim that the solution to the system:

$$\begin{aligned} &\frac{\pi}{2} + \varepsilon_2 + \left(K(1-r_0) + \left(1 + \frac{1}{r_0}\right) \max_j |\Omega_j| \right) t_e = \beta_\delta, \\ &2\varepsilon_2 = \left(\pi - \frac{\beta_\delta D(\Omega)}{K r_0 \sin\beta_\delta}\right) \left(K \frac{\sin\beta_\delta}{\beta_\delta} r_0 \right) t_e \end{aligned} \tag{3.2.20}$$

satisfies (3.2.19). Below, we will prove our claim.

Suppose that (t_e, ε_2) is a solution of system (3.2.20). Then, we use $D(\Theta_0) < \pi + 2\varepsilon_2$ to see that

$$\begin{aligned} 2\varepsilon_2 &= \left(\pi - \frac{\beta_\delta D(\Omega)}{K r_0 \sin\beta_\delta}\right) \left(K \frac{\sin\beta_\delta}{\beta_\delta} r_0 \right) t_e \\ &\Longrightarrow 2\varepsilon_2 < \left(\pi - \frac{\beta_\delta D(\Omega)}{K r_0 \sin\beta_\delta}\right) \left(e^{K \frac{\sin\beta_\delta}{\beta_\delta} r_0 t_e} - 1 \right) \\ &\Longrightarrow \pi + 2\varepsilon_2 + D(\Omega) \frac{\beta_\delta}{K r_0 \sin\beta_\delta} (e^{K \frac{\sin\beta_\delta}{\beta_\delta} r_0 t_e} - 1) < \pi e^{K \frac{\sin\beta_\delta}{\beta_\delta} r_0 t_e} \\ &\Longrightarrow \left(\pi + 2\varepsilon_2 + D(\Omega) \frac{\beta_\delta}{K \sin\beta_\delta r_0} (e^{K \frac{\sin\beta_\delta}{\beta_\delta} r_0 t_e} - 1) \right) e^{-K \frac{\sin\beta_\delta}{\beta_\delta} r_0 t_e} < \pi. \end{aligned} \tag{3.2.21}$$

On the other hand, it follows from (3.2.18) with $h = t_e$, (3.2.21) and $D(\Theta_0) < \pi + 2\varepsilon_2$ that we have the second inequality of (3.2.19):

$$\begin{aligned} D(\Theta(t_e)) &\le \left(D(\Theta_0) + D(\Omega) \frac{\beta_\delta}{K r_0 \sin\beta_\delta} (e^{K \frac{\sin\beta_\delta}{\beta_\delta} r_0 t_e} - 1) \right) e^{-K \frac{\sin\beta_\delta}{\beta_\delta} r_0 t_e} \\ &< \left(\pi + 2\epsilon + D(\Omega) \frac{\beta_\delta}{K r_0 \sin\beta_\delta} (e^{K \frac{\sin\beta_\delta}{\beta_\delta} r_0 t_e} - 1) \right) e^{-K \frac{\sin\beta_\delta}{\beta_\delta} r_0 t_e} \\ &< \pi. \end{aligned}$$

Note that system (3.2.20) admits the following solutions:

$$t_e = \frac{2\varepsilon_2}{\left(\pi - \frac{\beta_\delta D(\Omega)}{K r_0 \sin\beta_\delta}\right) \left(K \frac{\sin\beta_\delta}{\beta_\delta} r_0 \right)} \quad \text{and}$$

$$\varepsilon_2 \left(1 + \frac{\left(K(1-r_0) + \left(1 + \frac{1}{r_0}\right) |\Omega| \right)}{\left(\pi - \frac{\beta_\delta D(\Omega)}{K r_0 \sin\beta_\delta}\right) \left(K \frac{\sin\beta_\delta}{2\beta_\delta} r_0 \right)} \right) = \beta_\delta - \frac{\pi}{2}.$$

□

Before we state our second main result of this chapter, we recall several parameters to be used in the statement of main result. Let α and δ be positive constants in $(0, \frac{1}{2})$ satisfying the relation:

$$1 - \alpha(2 + \sin^2(1-\delta)\pi) > 0.$$

For notational simplicity, we set

$$\beta_\delta := (1-\delta)\pi, \qquad r_* := \frac{\max_j |\Omega_j|}{\sqrt{\alpha} K \sin\beta_\delta}, \qquad r^* := 1 - \alpha(2 + \sin^2\beta_\delta).$$

$$r_0 := r(0), \qquad \varepsilon_2(r_0, \delta, K, \{\Omega_i\}) := \frac{\beta_\delta - \frac{\pi}{2}}{1 + \frac{K(1-r_0) + \left(1 + \frac{1}{r_0}\right)\max_j |\Omega_j|}{\frac{\pi K r_0 \sin\beta_\delta}{2\beta_\delta} - \frac{D(\Omega)}{2}}}.$$

We now state our second main result on the complete frequency synchronization of nonidentical oscillators.

Theorem 3.2.2. *Suppose that the initial configuration Θ_0 and coupling strength K satisfy*

$$(i)\ r_* < r_0 \le r^*, \qquad \max_{1\le i\le N} |\theta_{i0} - \phi_0| < \frac{\pi}{2} + \varepsilon_2.$$

$$(ii)\ K > \max\left\{\frac{\max_j |\Omega_j|}{[1-\alpha(2+\sin^2\beta_\delta)][\sqrt{\alpha}\sin\beta_\delta]}, \frac{\beta_\delta D(\Omega)}{\pi r_0 \sin\beta_\delta}\right\}.$$

Then, the exponential frequency synchronization holds.

Proof. It follows from Lemma 3.2.3 that there exists $t_e \in (0, \infty)$ such that

$$D(\Theta(t_e)) < \pi.$$

We can apply Theorem 2.4.2 for the configuration at $t = t_e$ as a new initial configuration to derive the desired exponential frequency synchronization. □

Remark 3.2.2. *The conditions on initial configurations and coupling strength in Theorem 3.2.2 are not necessary conditions as can be seen in [33] where the relaxation toward the phase-locked state can be algebraic depending on the relation between the coupling strength and natural frequency diameter. Thus, our conditions on initial configurations and coupling strength are sufficient conditions to get the fast (exponential) frequency synchronization.*

Chapter 4

Kuramoto model on network with frustration

In this chapter, we study the dynamics of the Kuramoto model with pairwise interaction frustration on an all-to-all like network. Suppose that the interaction between the i-th and j-th oscillators are represented by the symmetric capacity matrix $\Psi = (\psi_{ij})$, where $\psi_{ij} \geq 0$, and let $\alpha_{ij} = \alpha_{ji}$ be a positive frustration. We study the dynamics of the Kuramoto model governed by the following system:

$$\begin{aligned} &\dot{\theta}_i = \Omega_i + \frac{K}{\psi_i} \sum_{j=1}^{N} \psi_{ji} \sin(\theta_j - \theta_i + \alpha_{ij}), \quad t > 0, \\ &\theta_i(0) = \theta_{i0}, \quad \psi_i := \sum_{j=1}^{N} \psi_{ji}. \end{aligned} \tag{4.0.1}$$

We assume the capacity matrix Ψ and the frustration matrix (α_{ij}) satisfy the following structure assumptions:

$$\left| \frac{\psi_{ji}}{\psi_i} - \frac{1}{N} \right| \leq \frac{\varepsilon_\psi}{N}, \quad |\alpha_{ij} - \alpha| \leq \varepsilon_\alpha, \quad \alpha \in (0, \frac{\pi}{2}), \quad \alpha + \varepsilon_\alpha < \frac{\pi}{2}, \tag{4.0.2}$$

where ε_ψ and ε_α are small nonnegative constants to be determined later. This chapter is base on the joint work in [37]

4.1 Previous results with uniform frustration

The results from [34, 52] are recalled by introducing a reference angle $D^\infty \in (0, \frac{\pi}{2})$, its dual angle $D_*^\infty \in (\frac{\pi}{2}, \pi)$, and critical coupling strength K_{ef}:

$$\sin D^\infty = \sin D_*^\infty = \frac{D(\Omega) + K \sin|\alpha|}{K}, \quad K_{ef} := \frac{D(\Omega)}{1 - \sin|\alpha|}.$$

Then, it is easy to see that, because

$$\sin D^\infty = \frac{D(\Omega)}{K} + \sin|\alpha| > \sin|\alpha|,$$

we have $D^\infty > |\alpha|$.

Theorem 4.1.1. *[34]*

Suppose that the parameters $D(\Omega)$, K, and α satisfy

$$D(\Omega) > 0, \quad K \geq K_{ef}, \quad 0 < D(\Theta^0) < D_2^\infty - |\alpha|.$$

Then, we have the following estimates:

1. *(Existence of a trapping set):*

$$\sup_{t \geq 0} D(\Theta(t)) \leq D_2^\infty - |\alpha|.$$

2. *(Time-evolution of $D(\Theta)$): The diameter $D(\Theta)$ satisfies Gronwall's inequality:*

$$\frac{d}{dt} D(\Theta(t)) \leq D(\Omega) + K \sin|\alpha| - K \sin(D(\Theta) + |\alpha|), \quad \text{a.e. } t \in (0, \infty).$$

3. *(Finite-time transition toward the smaller set): for any $0 < \varepsilon \ll 1$, there exists a $t_0 = t_0(\varepsilon) > 0$ such that*

$$D(\Theta(t)) \leq D_1^\infty - |\alpha| + \varepsilon, \quad t \geq t_0.$$

4. *(Exponential complete synchronization): for any $0 < \varepsilon \ll 1$ with $D_1^\infty + \varepsilon < \frac{\pi}{2}$, there exists a $t_0 > 0$ such that*

$$D(\dot\Theta(t_0))e^{-K(t-t_0)} \leq D(\dot\Theta(t)) \leq D(\dot\Theta(t_0))e^{-K\cos(D_1^\infty+\varepsilon)(t-t_0)}, \quad t \geq t_0.$$

Theorem 4.1.2. [52] *Suppose that the natural frequencies and coupling strength satisfy*

$$D(\Omega) > 0, \quad K \geq K_{ef},$$

and let $\Theta = (\theta_1, \ldots, \theta_N)$ *and* $\tilde{\Theta} = (\tilde{\theta}_1, \ldots, \tilde{\theta}_N)$ *be two global solutions with the initial configurations* Θ^0 *and* $\tilde{\Theta}^0$ *satisfying the conditions:*

$$0 < D(\Theta^0),\ D(\tilde{\Theta}^0) < D_2^\infty - |\alpha|.$$

Then, the following estimates hold.

1. *Asymptotic* ℓ^∞*-contraction holds in the sense that there exists a* $t_0 > 0$ *such that*

$$\|\Theta(t+s) - \tilde{\Theta}(t+s)\|_\infty \leq \|\Theta(t) - \tilde{\Theta}(t)\|_\infty, \quad \forall\, t, s \geq t_0.$$

2. *The resulting phase-locked states are unique:*

$$\lim_{t\to\infty} (\theta_i(t) - \theta_j(t)) = \lim_{t\to\infty} (\tilde{\theta}_i(t) - \tilde{\theta}_j(t)).$$

3. *If* $\Omega_i < \Omega_j$*, then there exists* $t_{ij}^* \geq 0$ *such that*

$$\theta_i(t) < \theta_j(t) \quad \textit{for all } t \geq t_{ij}^*.$$

Before we close this section, we show that the system (4.0.1) cannot be written as a gradient flow for some suitable $\mathcal{C}^2$-potential. Note that system (4.0.1) can be rewritten as

$$\dot{\theta}_i = \Omega_i + \frac{K}{N} \sum_{j=1}^N \Big[\sin(\theta_j - \theta_i)\cos\alpha + \cos(\theta_j - \theta_i)\sin\alpha\Big]. \tag{4.1.3}$$

Suppose that the system (4.1.3) can be rewritten as a gradient flow for some $\mathcal{C}^2$-potential, i.e., there exists a $\mathcal{C}^2$-function $V:\ \mathbb{R}^N \to \mathbb{R}$ such that $\dot{\Theta} = -\nabla_\Theta V(\Theta)$, i.e.,

$$\partial_{\theta_i} V(\Theta) = -\Omega_i - \frac{K}{N} \sum_{j=1}^N \Big[\sin(\theta_j - \theta_i)\cos\alpha + \cos(\theta_j - \theta_i)\sin\alpha\Big].$$

Because the potential V is $\mathcal{C}^2$, we have

$$\partial_{\theta_j}\partial_{\theta_i}V(\Theta) = \partial_{\theta_i}\partial_{\theta_j}V(\Theta), \quad 1 \le i \neq j \le N.$$

or equivalently,

$$\begin{aligned}\cos(\theta_j - \theta_i)\cos\alpha - \sin(\theta_j - \theta_i)\sin\alpha \\ = \cos(\theta_i - \theta_j)\cos\alpha - \sin(\theta_i - \theta_j)\sin\alpha \\ = \cos(\theta_j - \theta_i)\cos\alpha + \sin(\theta_j - \theta_i)\sin\alpha.\end{aligned}$$

This yields

$$2\sin\alpha = 0, \quad \text{i.e.,} \quad \alpha = 0,$$

which gives a contradiction. Therefore, we cannot apply the strong machinery of gradient flow to the system (4.0.1) to derive the complete synchronization for the generic initial data as was done in [38].

4.2 Dynamics of local order parameters

In this section, we study the dynamic behavior of the local order parameters, which will be crucial in our later analysis in Section 4.3 and 4.4.

4.2.1 Local order parameters

In this subsection, we study the local order parameters related to the system (4.0.1). For a given phase configuration $\Theta = (\theta_1, \cdots, \theta_N)$, we define Kuramoto's local order parameter r_i and ϕ_i:

$$r_i e^{\mathrm{i}\phi_i} := \frac{1}{\psi_i}\sum_{j=1}^{N}\psi_{ji}e^{\mathrm{i}\theta_j}. \tag{4.2.1}$$

For the all-to-all coupling case where $\psi_{ji} = 1$, the local order parameter coincides with the original Kuramoto order parameters r and ϕ [2]:

$$re^{\mathrm{i}\phi} := \frac{1}{N}\sum_{j=1}^{N}e^{\mathrm{i}\theta_j}, \qquad r_i = r, \qquad \phi_i = \phi, \qquad 1 \le i \le N.$$

Note that r_i satisfies

$$0 \le r_i = \frac{1}{\psi_i}\Big|\sum_{j=1}^{N} \psi_{ji} e^{\mathrm{i}\theta_j}\Big| \le \frac{1}{\psi_i}\sum_{j=1}^{N} \psi_{ji} = 1.$$

We divide (4.2.1) by $e^{\mathrm{i}\phi_i}$ on both sides and compare the real and imaginary parts to obtain

$$r_i = \sum_{j=1}^{N} \frac{\psi_{ji}}{\psi_i}\cos(\theta_j - \phi_i), \qquad 0 = \sum_{j=1}^{N} \frac{\psi_{ji}}{\psi_i}\sin(\theta_j - \phi_i). \tag{4.2.2}$$

Uniform frustration

For the uniform frustration $\alpha_{ji} = \alpha$, the system (4.0.1) becomes

$$\dot{\theta}_i = \Omega_i + \frac{K}{\psi_i}\sum_{j=1}^{N} \psi_{ji}\sin(\theta_j - \theta_i + \alpha). \tag{4.2.3}$$

We divide the defining relation (4.2.1) by $e^{\mathrm{i}(\theta_i - \alpha)}$ to obtain

$$r_i e^{\mathrm{i}(\phi_i - \theta_i + \alpha)} = \frac{1}{\psi_i}\sum_{j=1}^{N} \psi_{ji} e^{\mathrm{i}(\theta_j - \theta_i + \alpha)}.$$

If we compare the imaginary part of both sides, we have

$$r_i \sin(\phi_i - \theta_i + \alpha) = \frac{1}{\psi_i}\sum_{j=1}^{N} \psi_{ji}\sin(\theta_j - \theta_i + \alpha).$$

Thus, the system (4.2.3) can be rewritten into another form:

$$\dot{\theta}_i = \Omega_i + K r_i \sin(\phi_i - \theta_i + \alpha). \tag{4.2.4}$$

We further differentiate the relation (4.2.1) with respect to t to obtain

$$\dot{r}_i e^{\mathrm{i}\phi_i} + \mathrm{i} r_i e^{\mathrm{i}\phi_i}\dot{\phi}_i = \frac{1}{\psi_i}\sum_{j=1}^{N} \psi_{ji}\mathrm{i} e^{\mathrm{i}\theta_j}\dot{\theta}_j.$$

By multiplying with $e^{-\mathrm{i}\phi_i}$ on both sides, we get

$$\dot{r}_i + \mathrm{i}r_i\dot{\phi}_i = \frac{1}{\psi_i}\sum_{j=1}^{N}\psi_{ji}\mathrm{i}e^{\mathrm{i}(\theta_j-\phi_i)}\dot{\theta}_j.$$

We now compare and separate the real and imaginary parts to obtain

$$\dot{r}_i = -\sum_{j=1}^{N}\frac{\psi_{ji}}{\psi_i}\sin(\theta_j-\phi_i)\dot{\theta}_j, \qquad r_i\dot{\phi}_i = \sum_{j=1}^{N}\frac{\psi_{ji}}{\psi_i}\cos(\theta_j-\phi_i)\dot{\theta}_j. \tag{4.2.5}$$

Non-uniform frustration

We now return to the Kuramoto model (4.0.1) and simplify it in terms of r_i and ϕ_i:

$$\begin{aligned}\dot{\theta}_i &= \Omega_i + \frac{K}{\psi_i}\sum_{j=1}^{N}\psi_{ji}\sin(\theta_j-\theta_i+\alpha_{ji})\\ &= \Omega_i + Kr_i\sin(\phi_i-\theta_i+\alpha)+\mathcal{E}_i,\end{aligned} \tag{4.2.6}$$

where we used (4.2.4) and the small perturbation term $\mathcal{E}_i$ is given by the following relation:

$$\mathcal{E}_i := \frac{K}{\psi_i}\sum_{j=1}^{N}\psi_{ji}(\sin(\theta_j-\theta_i+\alpha_{ji})-\sin(\theta_j-\theta_i+\alpha)). \tag{4.2.7}$$

Next we use the small variation property of α_{ji} and ψ_{ji}:

$$\begin{aligned}&\frac{1-\varepsilon_\psi}{N} \le \frac{\psi_{ji}}{\psi_i} \le \frac{1+\varepsilon_\psi}{N} \quad \text{and}\\ &|\sin(\theta_j-\theta_i+\alpha_{ji})-\sin(\theta_j-\theta_i+\alpha)| \le |\alpha_{ji}-\alpha| \le \varepsilon_\alpha.\end{aligned} \tag{4.2.8}$$

Then, we can use (4.2.8) to obtain the estimate for $\mathcal{E}_i$ in (4.2.7):

$$|\mathcal{E}_i| \le K\sum_{j=1}^{N}\left(\frac{1}{N}+\frac{\varepsilon_\psi}{N}\right)\varepsilon_\alpha = K(1+\varepsilon_\psi)\varepsilon_\alpha. \tag{4.2.9}$$

4.2.2 Estimates of local order parameters

In this subsection, we study the dynamic variations of the local order parameters introduced in previous subsection.

Lemma 4.2.1. *Suppose that the network structure $\Psi = (\psi_{ji})$ is a small perturbation of the all-to-all coupling in the sense that*

$$\Big|\frac{\psi_{ji}}{\psi_i} - \frac{1}{N}\Big| \leq \frac{\varepsilon_\psi}{N},$$

and for a given $T \in (0, \infty]$, let $\Theta = (\theta_1, \cdots, \theta_N)$ be a solution to (4.0.1) *such that there exists a positive constant r_m such that*

$$\inf_{0 \leq t < T} \min_{1 \leq i \leq N} r_i(t) \geq r_m.$$

Then there exist a positive constant $\lambda_\infty = \lambda_\infty(r_m, \varepsilon_\psi)$ satisfying the following estimates:

$$\max_{1 \leq i,j \leq N} |r_i - r_j| \leq \lambda_\infty \varepsilon_\psi, \qquad \max_{1 \leq i,j \leq N} |\phi_i - \phi_j| \leq \lambda_\infty \varepsilon_\psi.$$

Proof. • (Estimate of r_i-variations): Consider the following ratio:

$$\frac{r_i e^{\mathrm{i}\phi_i}}{r_j e^{\mathrm{i}\phi_j}} = \frac{r_i}{r_j} e^{\mathrm{i}(\phi_i - \phi_j)} = \frac{\sum_{k=1}^N \frac{\psi_{ki}}{\psi_i} e^{\mathrm{i}\theta_k}}{\sum_{k=1}^N \frac{\psi_{kj}}{\psi_j} e^{\mathrm{i}\theta_k}}.$$

By taking absolute value on both sides, we have the upper and lower bounds for r_i/r_j:

$$\frac{r_i}{r_j} \leq \frac{\sum_{k=1}^N \big|\big(\frac{1}{N} + \frac{\varepsilon_\psi}{N}\big)\big|}{\sum_{k=1}^N \big|\big(\frac{1}{N} - \frac{\varepsilon_\psi}{N}\big)\big|} = \frac{1+\varepsilon_\psi}{1-\varepsilon_\psi} \quad \text{and} \quad \frac{r_i}{r_j} \geq \frac{\sum_{k=1}^N \big|\big(\frac{1}{N} - \frac{\varepsilon_\psi}{N}\big)\big|}{\sum_{k=1}^N \big|\big(\frac{1}{N} + \frac{\varepsilon_\psi}{N}\big)\big|} = \frac{1-\varepsilon_\psi}{1+\varepsilon_\psi}. \tag{4.2.10}$$

This yields

$$\frac{r_i}{r_j} - 1 \leq \frac{1+\varepsilon_\psi}{1-\varepsilon_\psi} - 1 = \frac{2\varepsilon_\psi}{1-\varepsilon_\psi}, \qquad \frac{r_i}{r_j} - 1 \geq \frac{1-\varepsilon_\psi}{1+\varepsilon_\psi} - 1 = -\frac{2\varepsilon_\psi}{1+\varepsilon_\psi}.$$

Hence, we have

$$\Big|\frac{r_i}{r_j} - 1\Big| \leq \frac{2\varepsilon_\psi}{1-\varepsilon_\psi}. \tag{4.2.11}$$

We use (4.2.10) and (4.2.11) to obtain

$$\frac{|r_i - r_j|}{r_i + r_j} = \frac{|r_i/r_j - 1|}{r_i/r_j + 1} \le \frac{\frac{2\varepsilon_\psi}{1-\varepsilon_\psi}}{\frac{2}{1+\varepsilon_\psi}} = \Big(\frac{1+\varepsilon_\psi}{1-\varepsilon_\psi}\Big)\varepsilon_\psi.$$

This yields

$$|r_i - r_j| \le (r_i + r_j)\Big(\frac{1+\varepsilon_\psi}{1-\varepsilon_\psi}\Big)\varepsilon_\psi \le 2\Big(\frac{1+\varepsilon_\psi}{1-\varepsilon_\psi}\Big)\varepsilon_\psi.$$

• (Estimate of ϕ_i-variations): By direct calculation, we have

$$\begin{aligned} |r_i e^{\mathrm{i}\phi_i} - r_j e^{\mathrm{i}\phi_j}| &= \left|\sum_{k=1}^N \left(\frac{\psi_{ki}}{\psi_i} - \frac{\psi_{kj}}{\psi_j}\right) e^{\mathrm{i}\theta_k}\right| \\ &\le \sum_{k=1}^N \left|\frac{\psi_{ki}}{\psi_i} - \frac{\psi_{kj}}{\psi_j}\right| |e^{\mathrm{i}\theta_k}| = \sum_{k=1}^N \frac{2\varepsilon_\psi}{N} = 2\varepsilon_\psi. \end{aligned}$$

On the other hand, by the law of cosines, we have

$$|r_i e^{\mathrm{i}\phi_i} - r_j e^{\mathrm{i}\phi_j}|^2 = r_i^2 + r_j^2 - 2r_i r_j \cos(\phi_i - \phi_j).$$

This yields

$$\begin{aligned} \cos(\phi_i - \phi_j) &= \frac{r_i^2 + r_j^2 - |r_i e^{\mathrm{i}\phi_i} - r_j e^{\mathrm{i}\phi_j}|^2}{2r_j r_j} \\ &\ge \frac{r_i^2 + r_j^2 - 4\varepsilon_\psi^2}{2r_i r_j} = \frac{(r_i - r_j)^2 + 2r_i r_j - 4\varepsilon_\psi^2}{2r_i r_j} \\ &\ge 1 - \frac{2\varepsilon_\psi^2}{r_i r_j} \ge 1 - \frac{2\varepsilon_\psi^2}{r_m^2}. \end{aligned} \tag{4.2.12}$$

Because ε_ψ is sufficiently small, we use (4.2.12) to obtain

$$\begin{aligned} \left(\frac{2}{\pi}|\phi_i - \phi_j|\right)^2 &\le \sin^2|\phi_i - \phi_j| = 1 - \cos^2|\phi_i - \phi_j| \\ &\le 1 - \left(1 - \frac{2\varepsilon_\psi^2}{r_m^2}\right)^2 = 4\frac{\varepsilon_\psi^2}{r_m^2}\Big(1 - \frac{\varepsilon_\psi^2}{r_m^2}\Big). \end{aligned}$$

This yields

$$|\phi_i - \phi_j| \le \Big(\frac{\pi}{r_m}\sqrt{1-\frac{\varepsilon_\psi^2}{r_m^2}}\Big)\varepsilon_\psi.$$

Finally, we set

$$\lambda_\infty(r_m, \varepsilon_\psi) := \max\Big\{2\Big(\frac{1+\varepsilon_\psi}{1-\varepsilon_\psi}\Big), \frac{\pi}{r_m}\sqrt{1-\frac{\varepsilon_\psi^2}{r_m^2}}\Big\}.$$

□

4.3 Complete synchronization with non-uniform frustration

In this section, we present the existence of complete synchronization for the systems (4.0.1) and (4.0.2).

In this subsection, we study the existence of a positively invariant set $\mathcal{A}$ and show that it attracts nearby points, i.e., we prove the existence of an attractor.

For a given phase configuration $\Theta = (\theta_1, \cdots, \theta_N) \in [0, \pi)^N$, we set

$$\theta_M := \max_{1\le i\le N} \theta_i \qquad \text{and} \qquad \theta_m := \min_{1\le i\le N} \theta_i.$$

The phase diameter $D(\Theta)$ and the diameter of the natural frequency $D(\Omega)$ are defined by

$$D(\Theta) := \theta_M - \theta_m \quad \text{and} \quad D(\Omega) := \max_{1\le i,j\le N} |\Omega_i - \Omega_j|.$$

Lemma 4.3.1. *For a configuration $\Theta = (\theta_1, \cdots, \theta_N) \in [0, \pi)^N$, there exists a function $\mathcal{R}_j$ such that $|\mathcal{R}_j| \le 8\varepsilon_\alpha$ and*

$$\sin(\theta_j-\theta_M+\alpha_{Mj})-\sin(\theta_j-\theta_m+\alpha_{mj}) \le \sin(\alpha+\varepsilon_\alpha)-\sin\Big(D(\Theta)+(\alpha+\varepsilon_\alpha)\Big)+\mathcal{R}_j.$$

Proof. Note that elementary properties of trigonometric functions yield

$$
\begin{aligned}
&\sin(\theta_j - \theta_M + \alpha_{Mj}) - \sin(\theta_j - \theta_m + \alpha_{mj}) \\
&\qquad = \Big(\sin(\theta_j - \theta_M) \cos \alpha_{Mj} + \cos(\theta_j - \theta_M) \sin \alpha_{Mj} \Big) \\
&\qquad - \Big(\sin(\theta_j - \theta_m) \cos \alpha_{mj} + \cos(\theta_j - \theta_m) \sin \alpha_{mj} \Big) \\
&\qquad = \Big(\sin(\theta_j - \theta_M) - \sin(\theta_j - \theta_m) \Big) \cos(\alpha + \varepsilon_\alpha) \\
&\qquad + \Big(\cos(\theta_j - \theta_M) - \cos(\theta_j - \theta_m) \Big) \sin(\alpha + \varepsilon_\alpha) + \mathcal{R}_j \\
&\qquad =: J_{11} + J_{12} + \mathcal{R}_j,
\end{aligned}
\tag{4.3.1}
$$

where the remainder term $\mathcal{R}_j$ is given by

$$
\begin{aligned}
\mathcal{R}_j &= \sin(\theta_j - \theta_M) \Big(\cos \alpha_{Mj} - \cos(\alpha + \varepsilon_\alpha) \Big) \\
&+ \cos(\theta_j - \theta_M) \Big(\sin \alpha_{Mj} - \sin(\alpha + \varepsilon_\alpha) \Big) \\
&- \sin(\theta_j - \theta_m) \Big(\cos \alpha_{mj} - \cos(\alpha + \varepsilon_\alpha) \Big) \\
&- \cos(\theta_j - \theta_m) \Big(\sin \alpha_{mj} - \sin(\alpha + \varepsilon_\alpha) \Big).
\end{aligned}
\tag{4.3.2}
$$

• Case A (Estimate of $\mathcal{R}_j$): By the defining condition (4.3.2) of $\mathcal{R}_j$ and the mean-value theorem, we have

$$
\begin{aligned}
|\mathcal{R}_j| &\le |\cos \alpha_{Mj} - \cos(\alpha + \varepsilon_\alpha)| + |\sin \alpha_{Mj} - \sin(\alpha + \varepsilon_\alpha)| \\
&+ |\cos \alpha_{mj} - \cos(\alpha + \varepsilon_\alpha)| + |\sin \alpha_{mj} - \sin(\alpha + \varepsilon_\alpha)| \\
&\le |\alpha_{Mj} - (\alpha + \varepsilon_\alpha)| + |\alpha_{Mj} - (\alpha + \varepsilon_\alpha)| \\
&+ |\alpha_{mj} - (\alpha + \varepsilon_\alpha)| + |\alpha_{mj} - (\alpha + \varepsilon_\alpha)| \\
&\le 8\varepsilon_\alpha.
\end{aligned}
\tag{4.3.3}
$$

• Case B (Estimate of J_{11}): We use

$$
\sin(\theta_j - \theta_M) \le \frac{\sin D(\Theta)}{D(\Theta)} (\theta_j - \theta_M), \qquad \sin(\theta_j - \theta_m) \ge \frac{\sin D(\Theta)}{D(\Theta)} (\theta_j - \theta_m),
$$

to obtain

$$
\begin{aligned}
J_{11} &\le \frac{\sin D(\Theta)}{D(\Theta)} \Big[(\theta_j - \theta_M) - (\theta_j - \theta_m) \Big] \cos(\alpha + \varepsilon_\alpha) \\
&= -\sin D(\Theta) \cos(\alpha + \varepsilon_\alpha).
\end{aligned}
\tag{4.3.4}
$$

• Case C (Estimate of J_{12}): We use

$$\cos(\theta_j - \theta_M) \le 1, \qquad \cos(\theta_j - \theta_m) \ge \cos D(\Theta)$$

to find

$$J_{12} \le (1 - \cos D(\Theta)) \sin(\alpha + \varepsilon_\alpha). \tag{4.3.5}$$

In (4.3.1), we combine (4.3.3), (4.3.4), and (4.3.5) to obtain the desired result. □

Assume that $\alpha < \frac{\pi}{2}, \varepsilon_\alpha$, and ε_ψ are sufficiently small such that

$$\Delta(\varepsilon_\alpha, \varepsilon_\psi, \alpha) := 8\varepsilon_\alpha + 2\varepsilon_\psi + \sin(\alpha + \varepsilon_\alpha) < 1. \tag{4.3.6}$$

For a given $D(\Omega), \alpha, \varepsilon_\alpha$ and ε_ψ satisfying (4.3.6), we define a coupling strength K_{ef}:

$$K_{ef} := \frac{D(\Omega)}{1 - \Delta(\varepsilon_\alpha, \varepsilon_\psi, \alpha)}, \quad \text{or equivalently} \quad \frac{D(\Omega)}{K_{ef}} = 1 - \Delta(\varepsilon_\alpha, \varepsilon_\psi, \alpha).$$

Then, for $D(\Omega)$ and K satisfying

$$D(\Omega) > 0, \quad K > K_{ef}, \tag{4.3.7}$$

we consider the trigonometric equation:

$$\sin x := \frac{D(\Omega)}{K} + \Delta(\varepsilon_\alpha, \varepsilon_\psi, \alpha), \quad x \in (0, \pi). \tag{4.3.8}$$

Note that the condition (4.3.7) guarantees that the R.H.S. of (4.3.8) is strictly smaller than 1 so that equation (4.3.8) has two distinct real roots:

$$0 < D_1^\infty < \frac{\pi}{2} < D_2^\infty < \pi.$$

Moreover, because $\varepsilon_\alpha, \varepsilon_\psi \ll 1$, D_i^∞ satisfies

$$\sin D_i^\infty \approx \frac{D(\Omega)}{K} + \sin\alpha.$$

We are now ready to define a set $\mathcal{A}$:

$$\mathcal{A} := \Big\{ \Theta \in [0, 2\pi)^N \ : \ D(\Theta) < D_2^\infty - (\alpha + \varepsilon_\alpha) \Big\}.$$

In the following lemma, we show that the set $\mathcal{A}$ is a positively invariant for the Kuramoto flow (4.0.1) under the condition (4.3.7).

Lemma 4.3.2. (Positive invariance of the set $\mathcal{A}$) *Suppose that the following conditions hold.*

1. *The network structure $\Psi = (\psi_{ji})$ and the frustration matrix (α_{ij}) are small perturbations of the Kuramoto flow:*

$$\left|\frac{\psi_{ji}}{\psi_i} - \frac{1}{N}\right| \le \frac{\varepsilon_\psi}{N}, \qquad |\alpha_{ji} - \alpha| \le \varepsilon_\alpha,$$

   where $\alpha \in \left(0, \frac{\pi}{2}\right)$ and $\varepsilon_\psi, \varepsilon_\alpha$ are sufficiently small to satisfy

$$\Delta(\varepsilon_\alpha, \varepsilon_\psi, \alpha) < 1.$$

2. *The natural frequencies and coupling strength satisfy*

$$D(\Omega) > 0, \quad K > K_{ef}.$$

Then, the set $\mathcal{A}$ is positively invariant under the Kuramoto flow (4.0.1)*:*

$$\Theta_0 \in \mathcal{A} \qquad \Longrightarrow \qquad \Theta(t) \in \mathcal{A},\ t \ge 0.$$

Proof. For the proof, we first show that the set $\mathcal{A}$ is invariant for a small time, and then we argue that the positive invariance of the set $\mathcal{A}$ can be continued to any length of time.

• Step A (the set $\mathcal{A}$ is invariant under the flow (4.0.1) in a small time interval): We define a set $\mathcal{T}$ and its supremum $T* \in [0, \infty]$:

$$\mathcal{T} := \{T \ge 0 \mid D(\Theta(t)) < D_2^\infty - (\alpha + \varepsilon_\alpha), \quad \forall t \in [0, T)\}, \quad T^* := \sup \mathcal{T}.$$

Because $D(\Theta_0) < D_2^\infty - (\alpha + \varepsilon_\alpha)$ and $D(\Theta(t))$ is a continuous function in t, there exists a $T > 0$ such that

$$D(\Theta(t)) < D_2^\infty - (\alpha + \varepsilon_\alpha), \quad \forall t \in [0, T).$$

Hence, the set $\mathcal{T}$ is not empty.

• Step B (the set $\mathcal{A}$ is invariant under the flow (4.0.1) for all times): We now claim that

$$T^* = \infty.$$

Suppose that it is not; i.e., that $T^* < \infty$. Then, we have

$$D(\Theta(T^*)) = \lim_{t\to T^*} D(\Theta(t)) = D_2^\infty - (\alpha + \varepsilon_\alpha). \tag{4.3.9}$$

Note that for $t \in [0, T^*)$, we use Lemma 4.3.1 to see

$$\begin{aligned}
&\frac{d}{dt}D(\Theta(t)) \\
&= \Omega_M - \Omega_m + K\sum_{j=1}^N \left(\frac{\psi_{jM}}{\psi_M}\sin(\theta_j - \theta_M + \alpha_{jM}) - \frac{\psi_{jm}}{\psi_m}\sin(\theta_j - \theta_m + \alpha_{jM}) \right) \\
&\le D(\Omega) + \frac{K}{N}\sum_{j=1}^N (\sin(\theta_j - \theta_M + \alpha_{jM}) - \sin(\theta_j - \theta_m + \alpha_{jM})) \\
&+ K\sum_{j=1}^N \left[\left(\frac{\psi_{jM}}{\psi_M} - \frac{1}{N} \right)\sin(\theta_j - \theta_M + \alpha_{jM}) + \left(\frac{\psi_{jm}}{\psi_m} - \frac{1}{N} \right)\sin(\theta_j - \theta_m + \alpha_{jm}) \right] \\
&\le D(\Omega) + \frac{K}{N}\sum_{j=1}^N (\sin(\theta_j - \theta_M + \alpha_{jM}) - \sin(\theta_j - \theta_m + \alpha_{jm})) + 2K\varepsilon_\psi \\
&\le D(\Omega) + K\Big[\sin(\alpha + \varepsilon_\alpha) - \sin\Big(D(\Theta(t)) + (\alpha + \varepsilon_\alpha) \Big) \Big] + (8\varepsilon_\alpha + 2\varepsilon_\psi)K.
\end{aligned} \tag{4.3.10}$$

Because $D(\Theta(t)) + \alpha + \varepsilon_\alpha < D_2^\infty < \pi$ for $t \in [0, T^*)$, we have

$$\sin\Big(D(\Theta) + (\alpha + \varepsilon_\alpha) \Big) \ge \frac{\sin D_2^\infty}{D_2^\infty}(D(\Theta(t)) + \alpha + \varepsilon_\alpha). \tag{4.3.11}$$

Then, we use (4.3.10) and (4.3.11) to obtain

$$\begin{aligned}
\frac{d}{dt}D(\Theta(t)) &\le D(\Omega) + K\Delta(\varepsilon_\alpha, \varepsilon_\psi, \alpha) - \frac{K\sin D_2^\infty}{D_2^\infty}(D(\Theta(t)) + \alpha + \varepsilon_\alpha) \\
&= K\sin D_2^\infty - \frac{K\sin D_2^\infty}{D_2^\infty}(D(\Theta(t)) + \alpha + \varepsilon_\alpha),
\end{aligned} \tag{4.3.12}$$

where we used the relation (4.3.8):

$$D(\Omega) + K\Delta(\varepsilon_\alpha, \varepsilon_\psi, \alpha) = K\sin D_2^\infty.$$

Because $\frac{d}{dt}D(\Theta(t)) = \frac{d}{dt}(D(\Theta(t))+\alpha+\varepsilon_\alpha)$, the differential inequality (4.3.12) becomes

$$\frac{d}{dt}(D(\Theta(t)) + \alpha + \varepsilon_\alpha) \le K \sin D_2^\infty - \frac{K \sin D_2^\infty}{D_2^\infty}(D(\Theta(t)) + \alpha + \varepsilon_\alpha).$$

Then, Gronwall's lemma yields

$$D(\Theta(t)) \le D_2^\infty - \alpha - \varepsilon_\alpha + \Big(D(\Theta_0) + \alpha + \varepsilon_\alpha - D_2^\infty\Big)e^{-\frac{K \sin D_2^\infty}{D_2^\infty}t}, \quad t \in [0, T^*).$$

By letting $t \to T^*$ and using the assumption $D(\Theta_0) + \alpha + \varepsilon_\alpha < D_2^\infty$, we have

$$D(\Theta(T^*)) < D(\Theta_0), \quad \text{i.e.,} \quad D(\Theta(T^*)) - (\alpha + \varepsilon_\alpha) < D_2^\infty - (\alpha + \varepsilon_\alpha),$$

which contradicts (4.3.9). Hence we have that

$$T^* = \infty, \qquad \text{and} \qquad \Theta(t) \in \mathcal{A}, \ t \ge 0.$$

$\square$

Proposition 4.3.1. (Entrance to a trapping regime) *Suppose that the following conditions hold.*

1. *The network structure $\Psi = (\psi_{ji})$ and frustration matrix (α_{ij}) are small perturbations of the Kuramoto flow:*

$$\Big|\frac{\psi_{ji}}{\psi_i} - \frac{1}{N}\Big| \le \frac{\varepsilon_\psi}{N}, \quad |\alpha_{ji} - \alpha| \le \varepsilon_\alpha,$$

   where $\alpha \in \Big(0, \frac{\pi}{2}\Big)$ and $\varepsilon_\psi, \varepsilon_\alpha$ are sufficiently small to satisfy

$$\Delta(\varepsilon_\alpha, \varepsilon_\psi, \alpha) < 1.$$

2. *The natural frequencies, coupling strength, and initial configuration satisfy*

$$D(\Omega) > 0, \quad K > K_{ef}, \quad 0 < D(\Theta_0) < D_2^\infty - (\alpha + \varepsilon_\alpha).$$

Let $\Theta = (\theta_1, \cdots, \theta_N)$ be a solution to system (4.0.1). Then, for any positive constant δ_0 such that $D_1^\infty + \delta_0 < \frac{\pi}{2}$, there exists a $t_0 = t_0(\delta_0) > 0$ such that

$$D(\Theta(t)) \le D_1^\infty - \alpha - \varepsilon_\alpha + \delta_0, \quad \textit{for} \quad t \ge t_0.$$

Proof. Consider the ordinary differential equation:

$$\dot{y} + K \sin y = D(\Omega) + K\Delta(\varepsilon_\alpha, \varepsilon_\psi, \alpha). \tag{4.3.13}$$

Our assumption about K implies that $y_* = D_1^\infty$ is an equilibrium solution for the equation (4.3.13). y_* is locally stable, because in the neighborhood of y_*,

$$\dot{y} = \begin{cases} < 0 & for \quad y > y_* \\ > 0 & for \quad y < y_*. \end{cases} \tag{4.3.14}$$

Moreover, for any initial value $y(0)$ with $0 < y(0) < D_2^\infty$, the trajectory $y(t)$ monotonically approaches y_*. Therefore, for any $\delta_0 > 0$ satisfying $D_1^\infty + \delta_0 < \frac{\pi}{2}$, there exists a time t_0 such that

$$|y(t) - y_*| < \delta_0, \qquad \forall t \geq t_0.$$

We combine this analysis of (4.3.13) and the comparison with (4.3.9) to find

$$D(\Theta(t)) + \alpha + \varepsilon_\alpha < D_1^\infty + \delta_0, \qquad \forall t \geq t_0.$$

That is,

$$D(\Theta(t)) < D_1^\infty - \alpha - \varepsilon_\alpha + \delta_0, \qquad \forall t \geq t_0.$$

□

We are now ready to prove the complete synchronization for an initial configuration whose diameter is strictly smaller than π. We define the frequency diameter by

$$D(\omega(t)) := \max_{1 \leq i,j \leq N} |\omega_i - \omega_j|, \quad \text{where} \quad \omega_i := \dot{\theta}_i.$$

Theorem 4.3.1. *Suppose that the following conditions hold.*

1. *The network structure $\Psi = (\psi_{ji})$ and frustration matrix (α_{ij}) are small perturbations of the Kuramoto flow:*

$$\left| \frac{\psi_{ji}}{\psi_i} - \frac{1}{N} \right| \leq \frac{\varepsilon_\psi}{N}, \quad |\alpha_{ji} - \alpha| \leq \varepsilon_\alpha,$$

where $\alpha \in \left(0, \frac{\pi}{2}\right)$ and $\varepsilon_\psi, \varepsilon_\alpha$ are sufficiently small to satisfy

$$\Delta(\varepsilon_\alpha, \varepsilon_\psi, \alpha) < 1.$$

2. The natural frequencies, coupling strength, and initial configuration satisfy

$$D(\Omega) > 0, \quad K > K_{ef}, \quad 0 < D(\Theta_0) < D_2^\infty - (\alpha + \varepsilon_\alpha).$$

Let $\Theta = (\theta_1, \cdots, \theta_N)$ *be a solution to the system* (4.0.1)*. Then, for any positive constant* δ_0 *such that* $D_1^\infty + \delta_0 < \frac{\pi}{2}$*, there exists a* $t_0 > 0$ *such that*

$$D(\omega(t_0))e^{-K(1+2\varepsilon_\psi)(t-t_0)} \leq D(\omega(t)) \leq D(\omega(t_0))e^{-K(\cos(D_1^\infty+\delta_0)-2\varepsilon_\psi)(t-t_0)}$$

for $t \geq t_0$*.*

Proof. It follows from Proposition 4.3.1 that there exists a $t_0 > 0$ such that

$$D(\Theta(t)) \leq D_1^\infty - \alpha - \varepsilon_\alpha + \delta_0 \leq D_1^\infty + \delta_0, \qquad \forall t \geq t_0.$$

• Case A (upper bound estimate): We differentiate the system (4.0.1) and

$$\left| \frac{\psi_{ji}}{\psi_i} - \frac{1}{N} \right| \leq \frac{\varepsilon_\psi}{N}$$

to obtain

$$\begin{aligned} \frac{d\omega_i}{dt} &= \frac{K}{N} \sum_{j=1}^N \cos(\theta_j - \theta_i + \alpha_{ji})(\omega_j - \omega_i) \\ &\quad + K \sum_{j=1}^N \left(\frac{\psi_{ji}}{\psi_i} - \frac{1}{N} \right) \cos(\theta_j - \theta_i + \alpha_{ji})(\omega_j - \omega_i). \end{aligned} \tag{4.3.15}$$

For a given $t \geq 0$, we set the indices i_t and j_t such that

$$D(\omega(t)) = \max_{1 \leq k,l \leq N} |\omega_k - \omega_l| =: \omega_{j_t} - \omega_{i_t}.$$

Then, we use (4.3.15) for such i_t and j_t to obtain

$$\begin{aligned} \frac{d}{dt} D(\omega) &= \frac{d}{dt}(\omega_{j_t} - \omega_{i_t}) \\ &= \frac{K}{N} \sum_{k=1}^N [\cos(\theta_k - \theta_{j_t} + \alpha_{kj_t})(\omega_k - \omega_{j_t}) - \cos(\theta_k - \theta_{i_t} + \alpha_{ki_t})(\omega_k - \omega_{i_t})] \end{aligned}$$

$$
\begin{aligned}
&+ K \sum_{k=1}^{N} \left(\frac{\psi_{kj_t}}{\psi_{j_t}} - \frac{1}{N} \right) \cos(\theta_k - \theta_{j_t} + \alpha_{kj_t})(\omega_k - \omega_{j_t}) \\
&- K \sum_{k=1}^{N} \left(\frac{\psi_{ki_t}}{\psi_{i_t}} - \frac{1}{N} \right) \cos(\theta_k - \theta_{i_t} + \alpha_{ki_t})(\omega_k - \omega_{i_t}) \\
&=: J_{21} + J_{22} + J_{23}. \qquad (4.3.16)
\end{aligned}
$$

Next, we estimate the terms $J_{2i},\ i = 1, \cdots, 3$ separately.

• Case A (Estimate of J_{21}): In this case, we use the result of Proposition 4.3.1:

$$
|\theta_k - \theta_i + \alpha_{ki}| \le D(\Theta(t)) + \alpha + \varepsilon_\alpha \le D_1^\infty + \delta_0, \quad t \ge t_0(\delta_0)
$$

and the facts that $\omega_k - \omega_{j_t} \le 0$ and $\omega_k - \omega_{i_t} \ge 0$ to obtain

$$
\begin{aligned}
J_{21} &\le \frac{K}{N} \sum_{k=1}^{N} [\cos(\theta_k - \theta_{j_t} + \alpha_{kj_t})(\omega_k - \omega_{j_t}) - \cos(\theta_k - \theta_{i_t} + \alpha_{ki_t})(\omega_k - \omega_{i_t})] \\
&\le \frac{K \cos(D_1^\infty + \delta_0)}{N} \sum_{k=1}^{N} [(\omega_k - \omega_{j_t}) - (\omega_k - \omega_{i_t})] \\
&= -K \cos(D_1^\infty + \delta_0) D(\omega), \qquad t \ge t_0(\delta).
\end{aligned} \qquad (4.3.17)
$$

• Case B (Estimate of J_{22} and J_{23}): We use $\left| \frac{\psi_{kj_t}}{\psi_{j_t}} - \frac{1}{N} \right|,\ \left| \frac{\psi_{ki_t}}{\psi_{i_t}} - \frac{1}{N} \right| \le \frac{\varepsilon_\psi}{N}$ to obtain

$$
|J_{2i}| \le K \varepsilon_\psi D(\omega), \quad i = 2, 3. \qquad (4.3.18)
$$

Finally, we combine (4.3.19) and (4.3.18) to obtain

$$
\frac{d}{dt} D(\omega) \le -K(\cos(D_1^\infty + \delta_0) - 2\varepsilon_\psi) D(\omega), \quad t \ge t_0.
$$

This gives us

$$
D(\omega(t)) \le D(\omega(t_0)) e^{-K(\cos(D_1^\infty + \delta_0) - 2\varepsilon_\psi)(t - t_0)}, \qquad t \ge t_0.
$$

- Case C (Lower bound estimate): In 4.3.16, we use $\cos\theta \le 1$ and (4.3.18) to get

$$\frac{d}{dt}D(\omega(t)) \ge -K(1+2\varepsilon_\psi)D(\omega(t)). \tag{4.3.19}$$

This gives us

$$D(\omega(t)) \ge D(\omega(t_0))e^{-K(1+2\varepsilon_\psi)(t-t_0)}, \qquad t \ge t_0.$$

□

In the following sections, we are going to extend the above result to a larger class of initial configurations, i.e., we will address an exponential complete phase synchronization of Kuramoto oscillators for some class of initial configurations, including the half circle.

4.4 Complete synchronization under reduced constraints

In this section, we study the synchronous dynamics of the Kuramoto oscillators for a larger class of initial configurations than in the previous section (the all-to-all network and uniform frustration), i.e.,

$$\psi_{ij} = 1, \quad \alpha_{ij} = \alpha, \quad 1 \le i, j \le N.$$

For a positive constant $\delta < \frac{1}{2}$, we set

$$\beta_\delta := (1-\delta)\pi.$$

Throughout this section, we will assume that α is small enough to satisfy

$$0 < \alpha < \min\{\frac{1}{4}, \quad \frac{\pi}{4}\sin\beta_\delta\Big[\frac{1}{\beta_\delta} - \frac{2\sqrt{2\bar{R}}}{(\beta_\delta - \frac{\pi}{2})\pi}\Big]\},$$

where $0 < \bar{R} < (1 - \frac{\pi}{2\beta_\delta})^2 < \frac{1}{2}$. Such a condition for α will appear in the Appendix. In this simplified setting, the system (4.0.1) becomes

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i + \alpha), \quad t > 0,\ 1 \le i \le N$$

and order parameters r and ϕ are defined by the relation

$$re^{\mathrm{i}\phi} = \frac{1}{N}\sum_{j=1}^{N} e^{\mathrm{i}\theta_j}.$$

Then, it follows from (4.2.2) that

$$r = \frac{1}{N}\sum_{j=1}^{N}\cos(\theta_j - \phi), \qquad 0 = \frac{1}{N}\sum_{j=1}^{N}\sin(\theta_j - \phi).$$

4.4.1 Estimate of the order parameters

In this subsection, we present some estimates on the local order parameters introduced in Section 4.2. For a given configuration $\Theta = (\theta_1, \cdots, \theta_N)$ with average phase ϕ, we define the index sets $I_\pm$ as follows.

$$I_+ := \Big\{ j : |\theta_j - \phi| < \frac{\pi}{2} \Big\}, \qquad I_- := \Big\{ j : \frac{\pi}{2} \le |\theta_j - \phi| \le \beta_\delta \Big\}. \tag{4.4.1}$$

Lemma 4.4.1. *Let $\Theta = (\theta_1, \cdots, \theta_N)$ be a configuration satisfying (4.0.1) and the relation:*

$$\theta_j - \phi \in [-\beta_\delta, \beta_\delta] \quad \textit{for } j = 1, 2, \ldots, N, \quad t \ge 0.$$

Then, we have

$$\frac{1}{N}\sum_{j=1}^{N}\sin^2(\theta_j - \phi) \ge R\sin^2\beta_\delta, \qquad R := \frac{1-r}{\sin^2\beta_\delta - 2\cos\beta_\delta}.$$

Proof. It follows from the first relation of (4.2.2) that

$$\begin{aligned}
&\frac{1}{N}\sum_{j=1}^{N}\cos(\theta_j - \phi) = r \\
&\qquad \Longleftrightarrow \frac{1}{N}\sum_{j\in I_+}\cos(\theta_j - \phi) = r - \frac{1}{N}\sum_{j\in I_-}\cos(\theta_j - \phi) \qquad\qquad (4.4.2)\\
&\qquad \Longrightarrow \sum_{j\in I_+}\cos(\theta_j - \phi) \le Nr - \sum_{j\in I_-}\cos\beta_\delta = Nr - |I_-|\cos\beta_\delta.
\end{aligned}$$

We use (4.4.2) to obtain

$$\begin{aligned}\sum_{j=1}^{N}\cos^2(\theta_j-\phi) &= \sum_{j\in I_+}\cos^2(\theta_j-\phi)+\sum_{j\in I_-}\cos^2(\theta_j-\phi)\\ &\le \sum_{j\in I_+}\cos(\theta_j-\phi)+\sum_{j\in I_-}\cos^2(\theta_j-\phi)\\ &\le Nr-|I_-|\cos\beta_\delta+|I_-|\cos^2\beta_\delta.\end{aligned}$$

This yields

$$\begin{aligned}\sum_{j=1}^{N}\sin^2(\theta_j-\phi) &= N-\sum_{j=1}^{N}\cos^2(\theta_j-\phi)\\ &\ge N-Nr+|I_-|(\cos\beta_\delta-\cos^2\beta_\delta)\\ &\ge N(1-r)+2|I_-|\cos\beta_\delta.\end{aligned}\tag{4.4.3}$$

We again use (4.4.3) to obtain

$$\frac{1}{N}\sum_{j=1}^{N}\sin^2(\theta_j-\phi)\ge\frac{N(1-r)+2|I_-|\cos\beta_\delta}{N}.\tag{4.4.4}$$

Let R be a number in $(0,1)$ to be determined later. Then, there are two cases categorized in terms of the ratio of $|I_-|$ to N.

- Case A ($|I_-|>RN$): We have

$$\frac{1}{N}\sum_{j=1}^{N}\sin^2(\theta_j-\phi)\ge\frac{1}{N}\sum_{j\in I_-}\sin^2(\theta_j-\phi)\ge R\sin^2\beta_\delta.\tag{4.4.5}$$

- Case B ($|I_-|\le RN$): We use the definition of r and (4.4.4) to obtain

$$\frac{1}{N}\sum_{j=1}^{N}\sin^2(\theta_j-\phi)\ge\frac{N(1-r)+2|I_-|\cos\beta_\delta}{N}\ge 1-r+2R\cos\beta_\delta.\tag{4.4.6}$$

Now, we choose R to satisfy

$$R\sin^2\beta_\delta=1-r+2R\cos\beta_\delta \quad\Longrightarrow\quad R=\frac{1-r}{\sin^2\beta_\delta-2\cos\beta_\delta}.$$

Then, it follows from (4.4.5) and (4.4.6) that we obtain the desired estimate.

$\square$

Lemma 4.4.2. *Let ϕ be the overall phase of the configuration $\Theta = \Theta(t)$ whose dynamics is governed by* (4.0.1)*. Then, we have*

$$|\dot{\phi}| \le K\Big[1 - r(1-\sin\alpha)\Big] + \frac{1}{r}\max_{1\le j\le N}|\Omega_j|, \quad t > 0.$$

Proof. It follows from (4.2.5) that

$$\begin{aligned}\dot{\phi} &= \frac{1}{rN}\sum_{j=1}^{N}\cos(\theta_j-\phi)\Big(\Omega_j - Kr\sin(\theta_j-\phi-\alpha)\Big)\\ &= \frac{1}{rN}\sum_{j=1}^{N}\Omega_j\cos(\theta_j-\phi) - \frac{K}{N}\sum_{j=1}^{N}\cos(\theta_j-\phi)\sin(\theta_j-\phi-\alpha)\\ &=: J_{31}+J_{32}.\end{aligned} \tag{4.4.7}$$

Below, we estimate the terms J_{31} and J_{32} separately.

• (Estimate of J_{31}): We use a rough bound for Ω_j and $\mathcal{E}_j$ to arrive at

$$|J_{31}| \le \frac{1}{rN}\sum_{j=1}^{N}|\Omega_j||\cos(\theta_j-\phi)| \le \frac{1}{r}\max_{1\le j\le N}|\Omega_j|. \tag{4.4.8}$$

• (Estimate of J_{32}): Below, we provide the upper and lower bounds for J_{32} to derive the desired estimate for $|J_{32}|$.

◇ Case A (Upper bound): We use the estimate (4.2.11) and (4.2.2) to obtain

$$\begin{aligned}J_{32} &= -\frac{K}{N}\sum_{j=1}^{N}\cos(\theta_j-\phi)\sin(\theta_j-\phi-\alpha)\\ &= -\frac{K}{N}\sum_{j=1}^{N}\underbrace{(\cos(\theta_j-\phi)-1)(\sin(\theta_j-\phi-\alpha)-1)}_{\ge 0} - \frac{K}{N}\sum_{j=1}^{N}\cos(\theta_j-\phi)\\ &\quad-\frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j-\phi-\alpha)+K\\ &\le -Kr - \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j-\phi-\alpha)+K\end{aligned}$$

$$
\begin{aligned}
&= -Kr - \frac{K}{N}\sum_{j=1}^{N}\Big(\sin(\theta_j-\phi)\cos\alpha - \cos(\theta_j-\phi)\sin\alpha\Big) + K \\
&= K\Big[1 - r(1-\sin\alpha)\Big].
\end{aligned} \tag{4.4.9}
$$

$\diamond$ Case B (Lower bound): Similar to Case A, we have

$$
\begin{aligned}
J_{32} &\geq -\frac{K}{N}\sum_{j=1}^{N}\cos(\theta_j-\phi)\sin(\theta_j-\phi-\alpha) \\
&= -\frac{K}{N}\sum_{j=1}^{N}\Big[\underbrace{(\cos(\theta_j-\phi)-1)(\sin(\theta_j-\phi-\alpha)+1)}_{\leq 0} \\
&\qquad\qquad - \cos(\theta_j-\phi) + \sin(\theta_j-\phi-\alpha) + 1\Big] \\
&\geq \frac{K}{N}\sum_{j=1}^{N}\cos(\theta_j-\phi) - \frac{K}{N}\sum_{j=1}^{N}[\sin(\theta_j-\phi)\cos\alpha - \cos(\theta_j-\phi)\sin\alpha] - K \\
&= -K\Big[1 - r(1+\sin\alpha)\Big].
\end{aligned} \tag{4.4.10}
$$

We now combine (4.4.9) and (4.4.10) to obtain the desired estimate:

$$
|J_{32}| \leq K\Big[1 - r(1-\sin\alpha)\Big]. \tag{4.4.11}
$$

Finally, in (4.4.7), we combine (4.4.8) and (4.4.11) to obtain the desired estimate. □

4.4.2 Evolution of the phase-diameter

In this subsection, we study the evolution of the phase-diameter $D(\Theta)$ under the a priori condition of fluctuations.

Lemma 4.4.3. *For a positive constant* $T \in (0,\infty]$, *let* $\Theta = (\theta_1,\cdots,\theta_N)$ *be a solution to* (4.2.6) *satisfying the a priori condition:*

$$
\beta_T := \max_{0\leq\tau\leq T}\max\{\theta_M(\tau)-\phi(\tau),\ \phi(\tau)-\theta_m(\tau)\} < \pi. \tag{4.4.12}
$$

Then, the phase-diameter $D(\Theta)$ satisfies the following lower and upper bounds: For $t \in (0,T)$, we have

$$
\begin{aligned}
&(i)\ D(\Theta(t)) \geq D(\Theta_0)\exp\Big[-K(\cos\alpha+\sin\alpha)\int_0^t r(s)ds\Big] \\
&\qquad -D(\Omega)\int_0^t \exp\Big[-K(\cos\alpha+\sin\alpha)\int_s^t r(\tau)d\tau\Big]ds. \\
&(ii)\ D(\Theta(t)) \leq D(\Theta_0)\exp\Big[-K(\cos\alpha\frac{\sin\beta_T}{\beta_T}-\sin\alpha)\int_0^t r(s)ds\Big] \\
&\qquad +D(\Omega)\int_0^t \exp\Big[-K(\cos\alpha\frac{\sin\beta_T}{\beta_T}-\sin\alpha)\int_s^t r(\tau)d\tau\Big]ds.
\end{aligned}
$$

Proof. (i) (Lower bound estimate): We use system (4.2.6) and (4.2.9) to derive

$$
\begin{aligned}
\dot{D}(\Theta) &= \dot{\theta}_M - \dot{\theta}_m \\
&= \Omega_M - \Omega_m - Kr\sin(\theta_M-\phi-\alpha) + Kr\sin(\theta_m-\phi-\alpha) \\
&\geq -D(\Omega) - Kr\big(\sin(\theta_M-\phi-\alpha)-\sin(\theta_m-\phi-\alpha)\big) \\
&= -D(\Omega) - Kr[(\sin(\theta_M-\phi)-\sin(\theta_m-\phi))\cos\alpha \\
&\quad - (\cos(\theta_M-\phi)-\cos(\theta_m-\phi))\sin\alpha] \\
&\geq -D(\Omega) - Kr[(\theta_M-\theta_m)\cos\alpha + (\theta_M-\theta_m)\sin\alpha] \\
&= -D(\Omega) - Kr(\cos\alpha+\sin\alpha)D(\Theta).
\end{aligned} \tag{4.4.13}
$$

Then, Gronwall's lemma for (4.4.13) yields the desired lower bound estimate for $D(\Theta)$.

(ii) (Upper bound estimate): Note that under the a priori condition (4.4.12), i.e.,

$$-\beta_T \leq \theta_m - \phi \leq 0 \leq \theta_M - \phi \leq \beta_T, \quad \text{for some } 0 < \beta_T < \pi,$$

we have

$$
\begin{aligned}
\sin(\theta_M-\phi)-\sin(\theta_m-\phi) &\geq \frac{\sin\beta_T}{\beta_T}(\theta_M-\phi) - \frac{\sin\beta_T}{\beta_T}(\theta_m-\phi) \\
&= \frac{\sin\beta_T}{\beta_T}(\theta_M-\theta_m).
\end{aligned} \tag{4.4.14}
$$

Then, we use (4.4.14) to obtain

$$
\begin{aligned}
\dot{D}(\Theta) &= \Omega_M - \Omega_m - Kr\Big[\cos\alpha\big(\sin(\theta_M - \phi) - \sin(\theta_m - \phi)\big) \\
&\quad - \sin\alpha\big(\cos(\theta_M - \phi) - \cos(\theta_m - \phi)\big)\Big] \\
&\le D(\Omega) - Kr\cos\alpha\frac{\sin\beta_T}{\beta_T}(\theta_M - \theta_m) + Kr\sin\alpha(\theta_M - \theta_m) \\
&\le D(\Omega) - KrD(\Theta)(\cos\alpha\frac{\sin\beta_T}{\beta_T} - \sin\alpha).
\end{aligned}
\tag{4.4.15}
$$

We use (4.4.15) and Gronwall's lemma to obtain the desired estimate. □

In the following section, we extend the synchronization estimate to initial configurations whose diameter is larger than π.

4.4.3 Dynamics of the order parameters

In this subsection, we study the dynamics of the order parameters r and ϕ introduced in Section 4.2. For notational simplicity, we set

$$A := \cos\alpha\frac{\sin\beta_\delta}{\beta_\delta} - \sin\alpha.$$

Lemma 4.4.4. *Suppose that the positive constants A, ε_θ, and t_e satisfy the following relations:*

$$
\begin{aligned}
&D_2^\infty - \alpha - \frac{D(\Omega)}{KAr_0} > 0 \quad \textit{and} \\
&\pi + 2\varepsilon_\theta = \left(D_2^\infty - \alpha - \frac{D(\Omega)}{KAr_0}\right)KAr_0t_e + D_2^\infty - \alpha.
\end{aligned}
\tag{4.4.16}
$$

Then, we have

$$\Big[\pi + 2\varepsilon_\theta + \frac{D(\Omega)}{KAr_0}\Big(e^{KAr_0t_e} - 1\Big)\Big]e^{-KAr_0t_e} < D_2^\infty - \alpha.$$

Proof. We use the defining condition (4.4.16) and the inequality $e^x - 1 > x$ for $x > 0$ to obtain

$$\pi + 2\varepsilon_\theta \;\; = \;\; \left(D_2^\infty - \alpha - \frac{D(\Omega)}{KAr_0}\right)KAr_0t_e + D_2^\infty - \alpha$$

$$
\begin{aligned}
&< \left[D_2^\infty - \alpha - \frac{D(\Omega)}{KAr_0}\right]\Big(e^{KAr_0t_e} - 1\Big) + D_2^\infty - \alpha \\
&= -\frac{D(\Omega)}{KAr_0}\Big(e^{KAr_0t_e} - 1\Big) + (D_2^\infty - \alpha)e^{KAr_0t_e},
\end{aligned}
$$

i.e., we have

$$
\pi + 2\varepsilon_\theta + \frac{D(\Omega)}{KAr_0}\Big(e^{KAr_0t_e} - 1\Big) < (D_2^\infty - \alpha)e^{KAr_0t_e}
$$

or equivalently

$$
\Big[\pi + 2\varepsilon_\theta + \frac{D(\Omega)}{KAr_0}\Big(e^{KAr_0t_e} - 1\Big)\Big]e^{-KAr_0t_e} < D_2^\infty - \alpha.
$$

□

For $\delta < \frac{1}{2}$ and $0 < \bar{R} < (1 - \frac{\pi}{2\beta_\delta})^2 < \frac{1}{2}$, we set the two positive constants r_* and r^* as follows.

$$
r_* := \frac{1}{\sqrt{\bar{R}}\sin\beta_\delta}\Big(\frac{\max_{1\le j\le N}|\Omega_j|}{K} + \alpha\Big), \qquad r^* := 1 - \bar{R}(\sin^2\beta_\delta - 2\cos\beta_\delta). \tag{4.4.17}
$$

We define K_2 as

$$
K_2 := \frac{\max_{1\le j\le N}|\Omega_j|}{\big[\sqrt{\bar{R}}\sin\beta_\delta(1 - \bar{R}(\sin^2\beta_\delta - 2\cos\beta_\delta)) - \alpha\big]}
$$

so that $r_* < r^*$ is guaranteed for $K > K_2$.

Lemma 4.4.5. *Suppose that the initial configuration and parameters δ, α, η, and K satisfy*

$$
\max\{\theta_M(0) - \phi(0), \phi(0) - \theta_m(0)\} \le \beta_\delta, \qquad K > K_2.
$$

Then the following assertions hold.

1. *If $r_* \le r_0 \le r^*$, then the order parameter r is in a non-decreasing mode at $t = 0$:*

$$
\dot{r}(0) \ge 0.
$$

2. If $r_* \le r_0$ and as long as

$$\max_{0\le s\le t} \max\{\theta_M(s)-\phi(s), \phi(s)-\theta_m(s)\} \le \beta_\delta,$$

we have

$$\min_{0\le s\le t} r(s) \ge \min\{r_0, r^*\}.$$

Proof. It suffices to show that, as long as $r_* \le r_0 \le r^*$, r is in a non-decreasing mode at $t = 0$,

$$\dot r(0) \ge 0. \tag{4.4.18}$$

It follows from (4.2.5) and (4.2.6) that we obtain

$$\begin{aligned} \dot r &= -\frac{1}{N}\sum_{j=1}^N \sin(\theta_j-\phi)[\Omega_j - Kr\sin(\theta_j-\phi-\alpha)] \\ &= -\frac{1}{N}\sum_{j=1}^N \Omega_j \sin(\theta_j-\phi) + \frac{Kr}{N}\sum_{j=1}^N \sin(\theta_j-\phi)\sin(\theta_j-\phi-\alpha) \\ &=: J_{41}+J_{42}. \end{aligned} \tag{4.4.19}$$

• Case A (Estimate of J_{41}): We use Cauchy-Schwarz' inequality to obtain

$$\begin{aligned} |J_{41}| &\le \frac{1}{N}\sum_{j=1}^N (\sqrt{N}|\Omega_j|)\cdot\Big(\frac{1}{\sqrt{N}}|\sin(\theta_j-\phi)|\Big) \\ &\le \frac{1}{N}\left(N\sum_{j=1}^N |\Omega_j|^2\right)^{\frac12}\left(\frac{1}{N}\sum_{j=1}^N \sin^2(\theta_j-\phi)\right)^{\frac12} \\ &\le \max_{1\le j\le N}|\Omega_j| \left(\frac{1}{N}\sum_{j=1}^N \sin^2(\theta_j-\phi)\right)^{\frac12}. \end{aligned} \tag{4.4.20}$$

• Case B (Estimate of J_{42}): We expand the term $\sin(\theta_j-\phi-\alpha)$ and use Cauchy-Schwarz's inequality to get

$$J_{42} = \frac{Kr}{N}\sum_{j=1}^N \sin(\theta_j-\phi)\sin(\theta_j-\phi-\alpha)$$

$$
\begin{aligned}
&= \frac{Kr}{N}\sum_{j=1}^{N}\sin^2(\theta_j-\phi) \\
&\quad + \underbrace{\frac{Kr}{N}\sum_{j=1}^{N}\sin(\theta_j-\phi)\Big(\sin(\theta_j-\phi-\alpha)-\sin(\theta_j-\phi)\Big)}_{=:J_{421}}
\end{aligned} \tag{4.4.21}
$$

Note that

$$
|J_{421}| \le \frac{Kr\alpha}{N}\sum_{j=1}^{N}|\sin(\theta_j-\phi)| \le Kr\alpha\Big(\frac{1}{N}\sum_{j=1}^{N}\sin^2(\theta_j-\phi)\Big)^{\frac{1}{2}}. \tag{4.4.22}
$$

In (4.4.19), we combine the estimates (4.4.20), (4.4.21), and (4.4.22) to obtain

$$
\begin{aligned}
\dot{r} &\ge \frac{Kr}{N}\sum_{j=1}^{N}\sin^2(\theta_j-\phi) - \max_{1\le j\le N}|\Omega_j|\left(\frac{1}{N}\sum_{j=1}^{N}\sin^2(\theta_j-\phi)\right)^{\frac{1}{2}} \\
&\quad - Kr\alpha\Big(\frac{1}{N}\sum_{j=1}^{N}\sin^2(\theta_j-\phi)\Big)^{\frac{1}{2}} \\
&= \Big(\frac{1}{N}\sum_{j=1}^{N}\sin^2(\theta_j-\phi)\Big)^{\frac{1}{2}}\Big\{Kr\Big[\Big(\frac{1}{N}\sum_{j=1}^{N}\sin^2(\theta_j-\phi)\Big)^{\frac{1}{2}}-\alpha\Big]-\max_{1\le j\le N}|\Omega_j|\Big\}.
\end{aligned}
$$

To obtain the desired estimate (4.4.18), we claim:

$$
r_0\Big[\Big(\frac{1}{N}\sum_{j=1}^{N}\sin^2(\theta_{j0}-\phi_0)\Big)^{\frac{1}{2}}-\alpha\Big]-\frac{\max_{1\le j\le N}|\Omega_j|}{K}\ge 0. \tag{4.4.23}
$$

Proof of (4.4.23): It follows from (4.2.2) with $\psi_{ji}=1$ and the assumption that

$$
\begin{aligned}
&r_0 = \frac{1}{N}\sum_{j=1}^{N}\cos(\theta_{j0}-\phi_0)\le r^* \\
&\quad\Longleftrightarrow \sum_{j\in I_+}\cos(\theta_{j0}-\phi_0)+\sum_{j\in I_-}\cos(\theta_{j0}-\phi_0)\le Nr^* \\
&\quad\Longleftrightarrow \sum_{j\in I_+}\cos(\theta_{j0}-\phi_0)\le Nr^*-\sum_{j\in I_-}\cos(\theta_{j0}-\phi_0) \\
&\quad\Longrightarrow \sum_{j\in I_+}\cos(\theta_{j0}-\phi_0)\le Nr^*-|I_-|\cos\beta_\delta,
\end{aligned} \tag{4.4.24}
$$

where $\phi_0 = \phi(0)$ and $I_\pm$ are index sets defined in (4.4.1). On the other hand, we also use (4.4.24) to obtain

$$\begin{aligned}\sum_{j=1}^{N}\cos^2(\theta_{j0}-\phi_0) &= \sum_{j\in I_+}\cos^2(\theta_{j0}-\phi_0)+\sum_{j\in I_-}\cos^2(\theta_{j0}-\phi_0)\\ &\le \sum_{j\in I_+}\cos(\theta_{j0}-\phi_0)+\sum_{j\in I_-}\cos^2(\theta_{j0}-\phi_0)\\ &\le Nr^* - |I_-|\cos\beta_\delta + |I_-|\cos^2\beta_\delta.\end{aligned}$$

We use the inequality $\cos\beta_\delta - \cos^2\beta_\delta \ge 2\cos\beta_\delta$ to derive

$$\begin{aligned}\sum_{j=1}^{N}\sin^2(\theta_{j0}-\phi_0) &= N-\sum_{j=1}^{N}\cos^2(\theta_{j0}-\phi_0)\\ &\ge N - Nr^* + |I_-|(\cos\beta_\delta - \cos^2\beta_\delta)\\ &\ge N - Nr^* + 2|I_-|\cos\beta_\delta.\end{aligned}$$

This yields

$$\begin{aligned}& r_0\Big(\frac{1}{N}\sum_{j=1}^{N}\sin^2(\theta_{j0}-\phi_0)\Big)^{\frac{1}{2}} - r_0\alpha\\ &\qquad \ge r_0\Big(1 - r^* + 2\frac{|I_-|}{N}\cos\beta_\delta\Big)^{\frac{1}{2}} - r_0\alpha.\end{aligned} \tag{4.4.25}$$

We choose $\bar{R}$ to be a positive constant satisfying the relation $0 < \bar{R} < (1-\frac{\pi}{2\beta_\delta})^2 < \frac{1}{2}$. Below, we consider two cases:

$$\text{Either } |I_-| > \bar{R}N, \quad \text{or} \quad |I_-| \le \bar{R}N.$$

• Case A ($|I_-| > \bar{R}N$): From our definition of r_*, we have

$$\begin{aligned}& r_0\big(\frac{1}{N}\sum_{j=1}^{N}\sin^2(\theta_{j0}-\phi)\big)^{\frac{1}{2}} - r_0\alpha\\ &\ge r_0\big(\frac{1}{N}\sum_{j\in I_-}\sin^2(\theta_{j0}-\phi)\big)^{\frac{1}{2}} - r_0\alpha\\ &\ge r_0\big(\frac{|I_-|}{N}\sin^2\beta_\delta\big)^{\frac{1}{2}} - r_0\alpha\end{aligned}$$

$$
\begin{aligned}
&\geq r_0\sqrt{\bar{R}}\sin\beta_\delta - r_0\alpha \\
&\geq r_*\sqrt{\bar{R}}\sin\beta_\delta - \alpha = \frac{\max_{1\leq j\leq N}|\Omega_j|}{K}. \qquad (4.4.26)
\end{aligned}
$$

In (4.4.26), the last equality is due to the defining relation of r_* in (4.4.17).

• Case B ($|I_-| \leq \bar{R}N$): We use the defining relations for $r_*,\ r^*$, and (4.4.25) to obtain

$$
\begin{aligned}
&r_0\Big(\frac{1}{N}\sum_{j=1}^N \sin^2(\theta_{j0}-\phi_0)\Big)^{\frac{1}{2}} - r_0\alpha \\
&\geq r_0\Big(1 - r^* + 2\frac{|I_-|}{N}\cos\beta_\delta\Big)^{\frac{1}{2}} - r_0\alpha \\
&\geq r_0\Big(1 - r^* + 2\bar{R}\cos\beta_\delta\Big)^{\frac{1}{2}} - r_0\alpha \qquad (4.4.27) \\
&= r_0\sqrt{\bar{R}}\sin\beta_\delta - r_0\alpha \\
&\geq r_*\sqrt{\bar{R}}\sin\beta_\delta - \alpha \\
&= \frac{\max_{1\leq j\leq N}|\Omega_j|}{K}.
\end{aligned}
$$

Then, it follows from (4.4.26) and (4.4.27) that we have the desired estimate (4.4.23). □

4.4.4 Emergence of complete synchronization

In this subsection, we finally provide the proof of complete synchronization for a large class of initial conditions using the key estimates obtained in the previous subsection.

Lemma 4.4.6. *Suppose that the natural frequency and initial configuration satisfy*

$$
D(\Omega) > 0, \quad r_* \leq r_0 \leq r^*, \quad \max_{1\leq i\leq N}|\theta_{i0} - \phi_0| < \frac{\pi}{2} + \varepsilon_\theta,
$$

where ε_θ will be determined later. Then, there exists a large positive coupling K_∞ such that for any solution $\Theta = (\theta_1, \cdots, \theta_N)$ of (4.0.1) with $K \geq K_\infty$, there exists a finite-time $t_e \in (0,\infty)$ satisfying

$$
D(\Theta(t_e)) < D_2^\infty - \alpha.
$$

Proof. We define a set $\mathcal{T}$ and its supremum T^*:

$$\mathcal{T} := \{t \in [0,\infty) \ : \ \max_{0\le\tau\le t} \max\{(\theta_M - \phi)(\tau), (\phi - \theta_m)(\tau)\} < \pi\}, \quad T^* := \sup \mathcal{T}.$$

• Step A (The set $\mathcal{T}$ is nonempty): It follows from the initial condition that we have

$$\max\{(\theta_M(0) - \phi_0, \phi_0 - \theta_m(0)\} < \frac{\pi}{2} + \varepsilon_\theta.$$

Then, by the continuity of $\theta_M - \phi$ and $\phi - \theta_m$, there exists a $t' > 0$ such that

$$\max_{0\le\tau\le t'} \max\{(\theta_M - \phi)(\tau), \ (\phi - \theta_m)(\tau)\} \le \beta_\delta, \quad t' \in \mathcal{T}.$$

• Step B (Controlling the phase-variations $\theta_M - \phi$ and $\phi - \theta_m$): Note that (4.0.1) and Lemma 4.4.2 imply that

$$\begin{aligned}
&\frac{d\theta_M}{dt} = \Omega_M - Kr\sin(\theta_M - \phi - \alpha),\\
&\frac{d\theta_m}{dt} = \Omega_m - Kr\sin(\theta_m - \phi - \alpha),\\
&|\dot{\phi}| \le K\Big[1 - r(1 - \sin\alpha)\Big] + \frac{1}{r}\max_{1\le j\le N}|\Omega_j|.
\end{aligned} \tag{4.4.28}$$

Then, equations (4.4.28) imply that

$$\begin{aligned}
\frac{d}{dt}(\theta_M - \phi) &\le \Omega_M - Kr\sin(\theta_M - \phi - \alpha) + K\Big[1 - r(1-\sin\alpha)\Big] + \frac{1}{r}\max_{1\le j\le N}|\Omega_j|\\
&= \Omega_M - Kr\Big(\sin(\theta_M - \phi)\cos\alpha - \cos(\theta_M - \phi)\sin\alpha\Big)\\
&+ K\Big[1 - r(1-\sin\alpha)\Big] + \frac{1}{r}\max_{1\le j\le N}|\Omega_j|\\
&\le \Omega_M + Kr\sin\alpha + K\Big[1 - r(1-\sin\alpha)\Big] + \frac{1}{r}\max_{1\le j\le N}|\Omega_j|\\
&\le K\Big[1 - r(1 - 2\sin\alpha)\Big] + \Big(\frac{1}{r} + 1\Big)\max_{1\le j\le N}|\Omega_j|\\
\frac{d}{dt}(\phi - \theta_m) &\le -\Omega_m + Kr\sin(\theta_m - \phi - \alpha) + K\Big[1 - r(1-\sin\alpha)\Big] + \frac{1}{r}\max_{1\le j\le N}|\Omega_j|\\
&\le K[1 - r(1-\sin\alpha)] + \Big(1 + \frac{1}{r}\Big)(\max_{1\le j\le N}|\Omega_j|)\\
&\le K[1 - r(1-2\sin\alpha)] + \Big(1 + \frac{1}{r}\Big)(\max_{1\le j\le N}|\Omega_j|).
\end{aligned} \tag{4.4.29}$$

Because $r_* \le r_0 \le r^*$, it follows from Lemma 4.4.5 (ii) that

$$r(t) \ge r_0, \quad 0 \le t \le t'. \tag{4.4.30}$$

Then, (4.4.29) and (4.4.30) imply that for $0 < h < t'$,

$$\begin{aligned}
&\max_{0\le t\le h} \max\{\theta_M(t) - \phi(t), \phi(t) - \theta_m(t)\} \\
&\qquad \le \frac{\pi}{2} + \varepsilon_\theta + \left[K[1 - r_0(1 - 2\sin\alpha)] + \Big(1 + \frac{1}{r_0}\Big)(\max_{1\le j\le N} |\Omega_j|)\right] h.
\end{aligned} \tag{4.4.31}$$

As long as

$$\frac{\pi}{2} + \varepsilon_\theta + \left[K(1 - r_0(1 - 2\sin\alpha)) + \Big(1 + \frac{1}{r_0}\Big)(\max_{1\le j\le N} |\Omega_j|)\right] h \le \beta_\delta < \pi, \tag{4.4.32}$$

we can use (4.4.15) to obtain

$$\begin{aligned}
\frac{dD(\Theta)}{dt} &= \dot{\theta}_M - \dot{\theta}_m \\
&= \Omega_M - \Omega_m - Kr\sin(\theta_M - \phi - \alpha) + Kr\sin(\theta_m - \phi - \alpha) \\
&\le D(\Omega) - KrD(\Theta)(\cos\alpha \frac{\sin\beta_\delta}{\beta_\delta} - \sin\alpha) \\
&= D(\Omega) - KArD(\Theta).
\end{aligned}$$

This yields

$$\begin{aligned}
D(\Theta(h)) &\le D(\Theta_0)e^{-KA\int_0^h r(s)ds} + D(\Omega)\int_0^h e^{-KA\int_s^h r(\tau)d\tau} ds \\
&\le D(\Theta_0)e^{-KAr_0 h} + D(\Omega)\int_0^h e^{-KAr_0(h-s)} ds \\
&\le \left[D(\Theta_0) + \frac{D(\Omega)}{KAr_0}\Big(e^{KAr_0 h} - 1\Big)\right] e^{-KAr_0 h}.
\end{aligned} \tag{4.4.33}$$

• Step C (Determination of t_e and ε_θ): It follows from (4.4.31), (4.4.32), and (4.4.33) that if we can determine t_e and ε_θ to satisfy

$$\begin{aligned}
&\frac{\pi}{2} + \varepsilon_\theta + \left[K[1 - r_0(1 - 2\sin\alpha)] + \Big(1 + \frac{1}{r_0}\Big)(\max_{1\le j\le N} |\Omega_j|)\right] t_e \le \beta_\delta, \\
&D(\Theta(t_e)) \le \left[D(\Theta_0) + \frac{D(\Omega)}{KAr_0}\Big(e^{KAr_0 t_e} - 1\Big)\right] e^{-KAr_0 t_e} < D_2^\infty - \alpha,
\end{aligned} \tag{4.4.34}$$

then we are done. For simplicity of presentation, we set

$$\begin{aligned} C_1 &:= K[1 - r_0(1 - 2\sin\alpha)] + \Big(1 + \frac{1}{r_0}\Big)(\max_{1\le j\le N} |\Omega_j|) \\ C_2 &:= KAr_0(D_2^\infty - \alpha - \frac{D(\Omega)}{KAr_0}) \\ &= KAr_0(D_2^\infty - \alpha) - D(\Omega). \end{aligned}$$

For the existence of $(\varepsilon_\theta, t_e)$ satisfying the system of inequalities (4.4.34), we consider a system of equalities for $(\varepsilon_\theta, t_e)$:

$$\varepsilon_\theta + C_1 t_e = \beta_\delta - \frac{\pi}{2}, \qquad 2\varepsilon_\theta - C_2 t_e = D_2^\infty - \alpha - \pi.$$

By direct calculation, we have

$$t_e = \frac{2\beta_\delta - D_2^\infty + \alpha}{2C_1 + C_2}, \qquad \varepsilon_\theta = \frac{1}{2C_1 + C_2}\Big[C_1(D_2^\infty - \alpha - \pi) + C_2(\beta_\delta - \frac{\pi}{2})\Big]. \tag{4.4.35}$$

Then, it follows from Appendix A that there exists a positive coupling strength K_1 such that if $K > K_1$, we have

$$t_e > 0 \quad \text{and} \quad \varepsilon_\theta > 0.$$

Next, we show that $(\varepsilon_\theta, t_e)$ given by the relation (4.4.35) satisfies (4.4.34)): For this, we apply Lemma 5.4 to obtain

$$\Big[\pi + 2\varepsilon_\theta + \frac{D(\Omega)}{KAr_0}\Big(e^{KAr_0 t_e} - 1\Big)\Big]\, e^{-KAr_0 t_e} < D_2^\infty - \alpha. \tag{4.4.36}$$

On the other hand, it follows from (4.4.33) with $h = t_e$, (4.4.36) and $D(\Theta_0) < \pi + 2\varepsilon_\theta$ that we have the second inequality of (4.4.34):

$$\begin{aligned} D(\Theta(t_e)) &\le \Big(D(\Theta_0) + \frac{D(\Omega)}{KAr_0}\Big(e^{KAr_0 t_e} - 1\Big)\Big)e^{-KAr_0 t_e} \\ &< \Big(\pi + 2\varepsilon_\theta + \frac{D(\Omega)}{KAr_0}\Big(e^{KAr_0 t_e} - 1\Big)\Big)e^{-KAr_0 t_e} \\ &\le D_2^\infty - \alpha. \end{aligned}$$

• Step C (Determination of K_∞): Because we use Lemma 5.2 to determine ε_θ and t_e, K has to satisfy the condition (4.4.16) :

$$D_2^\infty - \alpha - \frac{D(\Omega))}{KAr_0} > 0$$

From (4.3.8) and because $x > \sin x$ for $0 \le x \le \frac{\pi}{2}$, we attain

$$\begin{aligned}\frac{D(\Omega)}{K} + \sin\alpha = \sin D_2^\infty = \sin(\pi - D_2^\infty) < \pi - D_2^\infty \\ < \pi - \alpha - \frac{D(\Omega)}{KAr_0} < \pi - \alpha - \frac{D(\Omega)}{KA}\end{aligned}$$

Hence, we have

$$K > \frac{D(\Omega)(1+\frac{1}{A})}{\pi - \alpha - \sin\alpha} =: K_3.$$

By combining with (4.3.7), we now define

$$K_\infty := \max\{K_1, K_2, K_3\}.$$

Then, as long as $K > K_\infty$, there exists a finite positive time $t_e > 0$ such that

$$D(\Theta(t_e)) < D_2^\infty - \alpha.$$

Therefore, we restart our Kuramoto flow (4.0.1) with the new initial data $\Theta(t_e)$ in the time-interval $[t_e, \infty)$ and apply □

We now state our second main result.

Theorem 4.4.1. *Suppose that the initial configuration* Θ_0 *and coupling strength* K *satisfy*

$$(i)\ r_* \le r_0 \le r^*, \qquad \max_{1\le i\le N} |\theta_{i0} - \phi_0| < \frac{\pi}{2} + \varepsilon_\theta.$$
$$(ii)\ K > \max\{K_1, K_2, K_3, K_{ef}\}.$$

Then, the exponential synchronization holds:

$$\lim_{t\to\infty} \max_{1\le i,j\le N} |\dot\theta_i(t) - \dot\theta_j(t)| = 0.$$

Proof. It follows from Proposition 4.3.1 that there exists a $t_e \in (0, \infty)$ such that

$$D(\Theta(t_e)) < \pi.$$

We can apply Theorem 2.4.2 for the configuration at $t = t_e$ as a new initial configuration to derive the desired exponential synchronization. $\square$

Chapter 5

Kuramoto model with heterogeneous dynamics

The classical Kuramoto model (1.0.1) is consist of two terms; natural frequency and interactions between coupled oscillators. Since the intrinsic dynamics for the Kuramoto oscillator is governed by a constant natural frequency, the uncoupled Kuramoto oscillator has simple linear dynamics on the unit circle. When external fields affect to the dynamics, the intrinsic dynamics becomes rather complicated. For example, the daily life cycle and sleeping rhythm can be perturbed by the daylight. In this chapter, we consider the Kuramoto coupled system with complicated and heterogeneous intrinsic dynamics on the symmetric and connected network. Since the R.H.S. of (1.0.1) is 2π-periodic, (1.0.1) is a dynamical system on N-tori $\mathbb{S}^N$. However, we can lift the dynamical system as a dynamical system on the Euclidean space $\mathbb{R}^N$. Thus, the phase θ_i is assumed to take a value in $\mathbb{R}$ instead of taking mod 2π. In the absence of interaction between nodes, we assume the dynamics of θ_i follows

$$\dot{\theta}_i(t) = F_i(\theta_i, t), \tag{5.0.1}$$

where the external forcing $F_i : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ is given by a $\mathcal{C}^1$-function. Let $\Psi = (\psi_{ij})$ be the capacity matrix for the network, which satifies symmetry and connectedness:

(i) $\psi_{ij} = \psi_{ji} \geq 0, \qquad 1 \leq i, j \leq N,$

(ii) For any $(i, j) \in V \times V$, there is a shortest path from i to j, say

$$i = k_0 \to k_1 \to \cdots \to k_{d_{ij}} = j, \qquad (k_l, k_{l+1}) \in E, \quad l = 0, 1, \ldots, d_{ij} - 1. \tag{5.0.2}$$

We assume that the dynamics of θ_i is given by the following Kuramoto coupled system:

$$\begin{aligned} &\dot{\theta}_i = F_i(\theta_i, t) + K \sum_{j=1}^{N} \psi_{ij} \sin(\theta_j - \theta_i), \quad t > 0, \\ &\theta_i(0) = \theta_{i0}. \end{aligned} \tag{5.0.3}$$

With constant forcings and uniform all-to-all network structure, the sysetm (5.0.3) reduces to the classical Kuramoto model (1.0.1). This chapter is based the joint work in [41].

5.1 Practical synchronization and basic estimates

In this section, we study the concept of practical synchronization, and provide several basic estimates to be used in later sections. We first set the phase-diameter and energy as follows. For a configuration $\Theta = (\theta_1, \cdots, \theta_N) \in \mathbb{R}^N$, we set

$$\begin{aligned} &\theta_c := \frac{1}{N} \sum_{i=1}^{N} \theta_i, \quad \hat{\theta}_i := \theta_i - \theta_c, \quad D(\Theta) := \max_{1 \leq i,j \leq N} |\theta_i - \theta_j|, \\ &\mathcal{E}(\Theta) := \frac{1}{N} \sum_{i=1}^{N} |\theta_i|^2, \quad \mathcal{V}(\Theta) := \frac{1}{N} \sum_{i=1}^{N} |\hat{\theta}_i|^2. \end{aligned}$$

We define the concept of complete synchronization and practical synchronization for Kuramoto oscillators as follows.

Definition 5.1.1. *Let $\Theta = (\theta_1, \ldots, \theta_N)$ be a dynamical solution to system* (5.0.3)-(5.0.2).

1. *The dynamical solution* $\Theta = \Theta(t)$ *shows* asymptotic complete synchronization *if and only if the following condition holds:*

$$\lim_{t\to\infty}(D(\Theta(t)) + D(\dot{\Theta}(t))) = 0.$$

2. *The dynamical solution* $\Theta = \Theta(t)$ *shows* asymptotic practical synchronization *if and only if the following condition holds:*

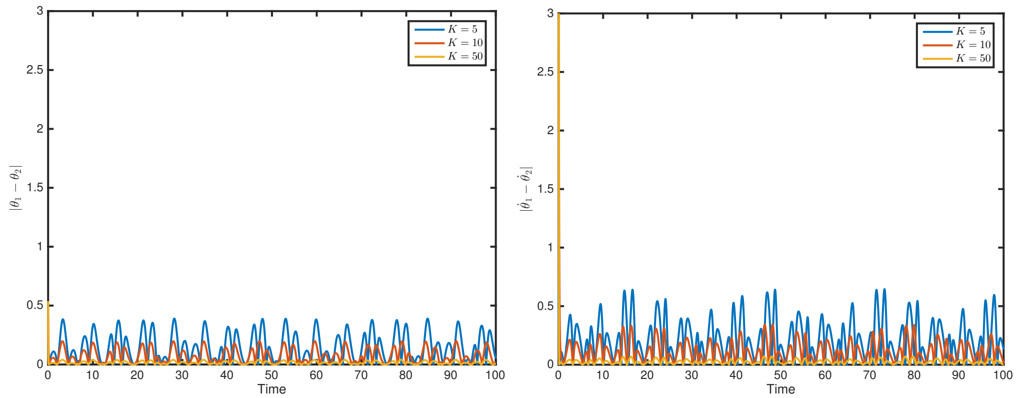
$$\lim_{K\to\infty}\limsup_{t\to\infty} D(\Theta(t)) = 0.$$

Remark 5.1.1. *1. In previous literature, the practical synchronization appeared in chaotic systems [11, 23, 29, 51, 53, 54] and in the first-order linear consensus model [47]. In Definition 5.1.1, we closely follows the stronger notion of practical synchronization from [47] saying that θ-differences can be made arbitrary small by suitable controls. Note that in our system* (5.0.3)*, the magnitude of control terms which are the sinusoidal couplings is dominated by the coupling strength K. In [11, 23, 29, 51, 53, 54, 67], the weaker concepts of practical synchronization comparing the Definition 5.1.1 were used to denote the uniform boundedness of phase differences in time but no restriction on the bound of the phase differences according to the coupling K. The numerical experiment of [67] shows that large coupling strength is necessary to obtain sufficiently small bound of phase diameter.*

2. For the study of synchronization phenomena of Kuramoto oscillators with intrinsic dynamics, complete synchronization cannot occur in general. To see this we consider the following two-oscillator system:

$$\frac{d\theta_1}{dt} = \sin(t-\theta_1) + \frac{K}{2}\sin(\theta_2-\theta_1),$$
$$\frac{d\theta_2}{dt} = \sin(2t-\theta_2) + \frac{K}{2}\sin(\theta_1-\theta_2).$$

Numerical simulation result in Figure 5.1 clearly shows that the differences $\theta_1-\theta_2$ and $\dot{\theta}_1-\dot{\theta}_2$ do not converges to zero so we cannot obtain complete synchronization in phase and frequency [19, 25, 31]. However, we can observe that the differences become smaller as coupling strength K is increased, in other words, this system is practically synchronized in the sense of Definition 5.1.1



(a) $|\theta_1 - \theta_2|$ (b) $|\dot{\theta}_1 - \dot{\theta}_2|$

Figure 5.1: Difference in state and rate of change for $K = 5,\ 10,\ 50$

Lemma 5.1.1. [32] *Suppose that the network $(V(G), E(G))$ is connected and the configuration $\Theta = (\theta_1, \cdots, \theta_N)$ has zero mean:*

$$\sum_{i=1}^{N} \theta_i = 0.$$

Then, we have

$$4L_* N^2 \mathcal{E}(\Theta) \le \sum_{i,j \in E(G)} |\theta_i - \theta_j|^2 \le 4N^2 \mathcal{E}(\Theta), \quad t \ge 0,$$

where the constant L_* is given by

$$L_* := \frac{1}{1 + \mathrm{diam}(G)|E^c(G)|}.$$

Here, E^c denotes the complement of the edge set E in $V \times V$ and $|E^c|$ denotes its cardinality.

Lemma 5.1.2. *Suppose that the phase configuration $\Theta = (\theta_1, \cdots, \theta_N) \in \mathbb{R}^N$ satisfies*

$$D(\Theta(t)) \le D_0 < \pi.$$

Then, we have

$$\sum_{(i,j) \in E} \psi_{ij}(\theta_j - \theta_i) \sin(\theta_j - \theta_i) \ge \frac{C^\infty}{D_0} \sum_{1 \le i,j \le N} |\hat{\theta}_j - \hat{\theta}_i|^2 = \frac{2N^2 C^\infty}{D_0} \mathcal{V}(\Theta).$$

In particular, if we have the additional zero sum condition $\sum_{i=1}^{N} \theta_i = 0$, *then*

$$\sum_{(i,j)\in E} \psi_{ij}(\theta_j - \theta_i)\sin(\theta_j - \theta_i) \geq \frac{2N^2 C^\infty}{D_0}\mathcal{E}(\Theta),$$

where the constant C^∞ *is defined by relation* (5.1.4).

$$C^\infty := L_* \psi_m \sin D_0, \qquad \psi_m := \min_{1\leq i,j\leq N} \psi_{ij}. \tag{5.1.4}$$

Proof. We use the following elementary inequality

$$x\sin x \geq \frac{\sin D_0}{D_0}x^2 \quad \text{for} \quad x \in [-D_0, D_0] \quad \text{and} \qquad \sum_{i=1}^{N} \hat{\theta}_i = 0,$$

to obtain

$$\begin{aligned}
&\sum_{(i,j)\in E} \psi_{ji}(\theta_j - \theta_i)\sin(\theta_j - \theta_i) \\
&\geq \frac{\sin D_0}{D_0} \sum_{(i,j)\in E} \psi_{ji}|\hat{\theta}_j - \hat{\theta}_i|^2 \\
&\geq \frac{\sin D_0}{D_0}\psi_m \sum_{(i,j)\in E} |\hat{\theta}_j - \hat{\theta}_i|^2 \\
&\geq \frac{\sin D_0}{D_0}\psi_m L_* \sum_{1\leq i,j\leq N} |\hat{\theta}_j - \hat{\theta}_i|^2 \\
&= \frac{\sin D_0}{D_0}\psi_m L_* \Big[\sum_{i,j=1}^{N} |\hat{\theta}_j|^2 - 2\Big(\sum_{i=1}^{N}\hat{\theta}_i\Big)\Big(\sum_{j=1}^{N}\hat{\theta}_j\Big) + \sum_{i,j=1}^{N}|\hat{\theta}_i|^2\Big] \\
&= \frac{2\sin D_0}{D_0}\psi_m L_* N \sum_{j=1}^{N} |\hat{\theta}_j|^2.
\end{aligned}$$

Here, the third inequality uses Lemma 5.1.1. □

Lemma 5.1.3. *For* $T \in (0,\infty]$, *let* $\Theta = \Theta(t)$ *be the solution to system* (5.0.3) *satisfying the a priori condition on the time-interval* $[0, T)$*:*

$$\sup_{t\in[0,T)} D(\Theta(t)) \leq D_0 < \pi, \quad \Delta\mathcal{F} := \sup_{i,\theta,t}\Big(\frac{\partial F_i}{\partial\theta}(\theta, t)\Big) < \infty.$$

Then, the variance $\mathcal{V}(\Theta)$ satisfies

$$\frac{d\mathcal{V}(\Theta)}{dt} \leq D(\mathcal{F})\sqrt{2}\sqrt{\mathcal{V}(\Theta)} - 2\Big[\frac{KNC^{\infty}}{D_0} - \Delta\mathcal{F}\Big]\mathcal{V}(\Theta), \quad t \in [0,T),$$

where $D(\mathcal{F})$ is the diameter of the family of forcing terms $\{F_1, \cdots, F_N\}$:

$$D(\mathcal{F}) := \max_{1\leq i,j\leq N} \|F_i - F_j\|_{L^{\infty}(\mathbb{R}\times\mathbb{R}_+)}.$$

Proof. We first note that the average θ_c and its perturbation $\hat{\theta}_i := \theta_i - \theta_c$ satisfy

$$\begin{aligned} \dot{\theta}_c &= \frac{1}{N}\sum_{i=1}^{N} F_i(\theta_i, t), \\ \dot{\hat{\theta}}_i &= \frac{1}{N}\sum_{j=1}^{N}\Big(F_i(t,\theta_i) - F_j(t,\theta_j)\Big) + K\sum_{j=1}^{N}\psi_{ji}\sin(\hat{\theta}_j - \hat{\theta}_i). \end{aligned} \tag{5.1.5}$$

We multiply the second equation of (5.1.5) by $2\hat{\theta}_i$, sum with respect to i, and divide by N to find

$$\begin{aligned} \frac{d\mathcal{V}(\Theta)}{dt} &= \frac{2}{N^2}\sum_{i,j=1}^{N}\hat{\theta}_i\big(F_i(\theta_i,t) - F_j(\theta_j,t)\big) + \frac{2K}{N}\sum_{i,j=1}^{N}\psi_{ji}\hat{\theta}_i\sin(\hat{\theta}_j - \hat{\theta}_i) \\ &= \frac{1}{N^2}\sum_{i,j=1}^{N}(\hat{\theta}_i - \hat{\theta}_j)\big(F_i(\theta_i,t) - F_j(\theta_j,t)\big) \\ &\quad - \frac{K}{N}\sum_{i,j=1}^{N}\psi_{ji}(\hat{\theta}_j - \hat{\theta}_i)\sin(\hat{\theta}_j - \hat{\theta}_i) \\ &=: \mathcal{I}_{11} + \mathcal{I}_{12}. \end{aligned} \tag{5.1.6}$$

Here, we used the symmetry of the network, $\psi_{ij} = \psi_{ji}$, and the trick $i \leftrightarrow j$. We now consider the two terms separately.

• (Estimate of $\mathcal{I}_{11}$): There exists θ^*_{ij} on the segment between θ_i and θ_j such that

$$\begin{aligned} F_i(\theta_i,t) - F_j(\theta_j,t) &= F_i(\theta_i,t) - F_j(\theta_i,t) + F_j(\theta_i,t) - F_j(\theta_j,t) \\ &= F_i(\theta_i,t) - F_j(\theta_i,t) + \frac{\partial F_j}{\partial\theta}(\theta^*_{ij},t)(\hat{\theta}_i - \hat{\theta}_j), \end{aligned} \tag{5.1.7}$$

where we used the fact that $\theta_i - \theta_j = \hat{\theta}_i - \hat{\theta}_j$.

Note that

$$
\begin{aligned}
\mathcal{I}_{11} &= \frac{1}{N^2}\sum_{i,j=1}^{N}(\hat{\theta}_i - \hat{\theta}_j)\big(F_i(\theta_i,t) - F_j(\theta_j,t)\big) \\
&= \frac{1}{N^2}\sum_{i,j=1}^{N}(\hat{\theta}_i - \hat{\theta}_j)\big(F_i(\theta_i,t) - F_j(\theta_i,t)\big) + \frac{1}{N^2}\sum_{i,j=1}^{N}\frac{\partial F_j}{\partial\theta}(\theta_{ij}^*,t)(\hat{\theta}_i - \hat{\theta}_j)^2 \\
&\le \frac{1}{N^2}D(\mathcal{F})\sum_{i,j=1}^{N}|\hat{\theta}_i - \hat{\theta}_j| + \frac{1}{N^2}\sup_{j,\theta,t}\Big(\frac{\partial F_j}{\partial\theta}(\theta,t)\Big)\sum_{i,j=1}^{N}|\hat{\theta}_i - \hat{\theta}_j|^2 \\
&\le D(\mathcal{F})\sqrt{2\mathcal{V}(\Theta)} + 2\Delta\mathcal{F}\mathcal{V}(\Theta),
\end{aligned}
$$

where we used (5.1.7) and

$$
\begin{aligned}
&\sum_{i,j=1}^{N}|\hat{\theta}_i - \hat{\theta}_j|^2 = 2N^2\mathcal{V}(\Theta), \\
&\sum_{i,j=1}^{N}|\hat{\theta}_i - \hat{\theta}_j| \le N\Big(\sum_{i,j=1}^{N}|\hat{\theta}_i - \hat{\theta}_j|^2\Big)^{\frac{1}{2}} \le N^2\sqrt{2\mathcal{V}(\Theta)}.
\end{aligned}
$$

• (Estimate of $\mathcal{I}_{12}$): It follows from Lemma 5.1.2 that we have

$$
\mathcal{I}_{12} \le -\frac{2KNC^\infty}{D_0}\mathcal{V}(\Theta). \tag{5.1.8}
$$

We combine (5.1.7)-(5.1.8) together with (5.1.6) to obtain the desired estimate. □

Remark 5.1.2. *Note that for an all-to-all coupling with $\psi_{ij} = \frac{1}{N}$, we have*

$$
NC^\infty = \sin D_0.
$$

Lemma 5.1.4. *For $T \in (0,\infty]$, let $\Xi = (\xi_1,\cdots,\xi_N)$ and $\Theta = (\theta_1,\cdots,\theta_N)$ be the corresponding solutions to decoupled system* (5.0.1) *and coupled system* (5.0.3) *with the same initial data Θ_0 and satisfying the following a priori conditions:*

1. *The total energy for the decoupled system is bounded:*

$$\sup_{0\le t<T} \mathcal{E}(\Xi) = \mathcal{E}^\infty(\Xi, T) < \infty. \tag{5.1.9}$$

2. *The phase-diameter is confined to the half-circle region: there exists $D_0 \in (0, \pi)$ such that*

$$\sup_{t\in[0,T)} D(\Theta(t)) \le D_0.$$

Then, the coupled solution $\Theta = \Theta(t)$ is bounded in the interval $[0, T)$, i.e., there exists $\theta^\infty(N, T) \in [0, \infty)$ such that

$$\sup_{0\le t<T} \max_{1\le i\le N} |\theta_i(t)| \le \theta^\infty(N, T).$$

Proof. We will show that the energy of the coupled system is smaller than that of the decoupled system. Then, by the framework of (5.1.9), the energy for the decoupled system is bounded, and we can derive the desired result. For the boundedness of $\mathcal{E}(\Theta)$, we multiply (5.0.3) by $2\theta_i$ and sum with respect to i to obtain

$$\begin{aligned}
\frac{d}{dt}\sum_{i=1}^N \theta_i^2 &= 2\sum_{i=1}^N \theta_i F_i(\theta_i, t) + 2K\sum_{i,j} \psi_{ji}\theta_i \sin(\theta_j - \theta_i) \\
&= 2\sum_{i=1}^N \theta_i F_i(\theta_i, t) - K\sum_{i,j=1}^N \psi_{ji}(\theta_j - \theta_i)\sin(\theta_j - \theta_i) \\
&\le 2\sum_{i=1}^N \theta_i F_i(\theta_i, t),
\end{aligned}$$

where we used Lemma 5.1.2 to find

$$\sum_{i,j=1}^N \psi_{ji}(\theta_j - \theta_i)\sin(\theta_j - \theta_i) \ge 0.$$

We now consider the solution to the decoupled system $\Xi = \Xi(t)$ with the same initial data:

$$\dot{\xi}_i = F_i(\xi_i, t), \ t > 0, \qquad \xi_i(0) = \theta_{i0}.$$

To obtain the time derivative of the energy functional of decoupled system $\mathcal{E}(\Xi)$, we estimate

$$\frac{d}{dt}\sum_{i=1}^{N}\xi_i^2 = 2\sum_{i=1}^{N}\xi_i F_i(\xi_i, t)$$

Then, by the comparison principle of ODEs, we have

$$\mathcal{E}(\Theta) := \frac{1}{N}\sum_{i=1}^{N}\theta_i^2(t) \le \frac{1}{N}\sum_{i=1}^{N}\xi_i^2(t) = \mathcal{E}(\Xi), \quad t \in [0, T).$$

Then, the a priori condition (5.1.9) yields

$$|\theta_i(t)| \le \sqrt{N\mathcal{E}(\Theta)} \le \sqrt{N\mathcal{E}(\Xi)} \le \sqrt{N\mathcal{E}^\infty(\Xi, T)} < \infty.$$

$\square$

5.2 Practical synchronization with heterogeneous forcing

In this section, we present several sufficient conditions for practical synchronization in terms of initial configurations, parameters and forcing terms. we consider heterogeneous forcing terms:

$$\text{There exists a pair} \quad i \ne j \quad \text{such that} \quad F_i \ne F_j.$$

We adopt the following framework $\mathcal{A}$ on the family $\mathcal{F} = \{F_1, \cdots, F_N\}$ of forcing and network structure Ψ and the coupling strength K.

($\mathcal{A}1$) $D(\mathcal{F}) := \max_{i,j} \|F_i - F_j\|_{L^\infty(\mathbb{R}\times\mathbb{R}_+)} < \infty, \quad \Delta\mathcal{F} := \sup_{i,\theta,t}\left(\frac{\partial F_i}{\partial \theta}(\theta, t)\right) < \infty,$

($\mathcal{A}2$) $K > \dfrac{D(\mathcal{F}) + \frac{D_0}{\sqrt{N}}\Delta\mathcal{F}}{\sqrt{N}C^\infty}.$

($\mathcal{A}3$) The initial configuration satisfies the following boundedness condition:

$$D(\theta_0) < D_0 < \pi, \qquad \mathcal{V}(\theta_0) < \frac{D_0^2}{2N}.$$

Theorem 5.2.1. *Suppose that the frameworks $\mathcal{A}$ holds, then practical synchronization is achieved:*

$$\lim_{K\to\infty}\limsup_{t\to\infty} D(\theta(t)) = 0.$$

5.2.1 Bounded forcing

We first consider the forcings $\mathcal{F} = \{F_1, \cdots, F_N\}$ satisfying the following boundedness conditions:

$$D(\mathcal{F}) = \max_{i,j} \|F_i - F_j\|_{L^\infty(\mathbb{R}\times\mathbb{R}_+)} < \infty, \qquad \Delta\mathcal{F} < \infty. \tag{5.2.10}$$

Note that the following forcings satisfy the boundedness condition (5.2.10):

$$F_i(\theta_i, t) = \Omega_i, \qquad F_i(\theta_i, t) = A_i \sin(\sigma t - \theta_i).$$

Lemma 5.2.1. *Suppose that the framework $\mathcal{A}$ holds. Then, the phase-diameter $D(\Theta(t))$ is uniformly bounded by D_0, i.e.,*

$$\sup_{0\le t<\infty} D(\Theta(t)) \le D_0.$$

Proof. We define

$$\mathcal{T} := \{T : D(\Theta(t)) < D_0,\ \forall t \in [0, T)\} \quad \text{and} \quad T_* := \sup \mathcal{T},$$

and claim that

$$T_* = \infty.$$

Proof of claim. We split the proof into two parts. In Step A, we show that the set $\mathcal{T}$ is nonempty, and in Step B, we show that $T_* = \infty$ using the differential inequality obtained in Lemma 5.1.3.

- **(Step A).** By the continuity of $D(\Theta(t))$, there exists a $\delta > 0$ such that

$$D(\Theta(t)) < D_0, \quad t \in [0, \delta), \quad i, j = 1, 2, \ldots, N.$$

Therefore, $\delta \in \mathcal{T}$ and $\mathcal{T} \neq \emptyset$.

- **(Step B).** Suppose not, i.e., $T_* < \infty$. Then, we should have

$$D(\Theta(t)) < D_0, \quad t \in [0, T_*), \qquad \lim_{t\to T_*-} D(\Theta(t)) = D_0. \tag{5.2.11}$$

We again use Gronwall's inequality in Lemma 5.1.3:

$$\frac{d\mathcal{V}(\Theta)}{dt} \le D(\mathcal{F})\sqrt{2\mathcal{V}(\Theta)} - 2\Big[\frac{KNC^\infty}{D_0} - \Delta\mathcal{F}\Big]\mathcal{V}(\Theta) \quad \text{on} \quad [0,T). \tag{5.2.12}$$

Note that the condition on K guarantees that the coefficient of the second term on the r.h.s. of (5.2.12) is positive.

$$Y := \sqrt{\mathcal{V}(\Theta)}, \quad t \ge 0.$$

Then, it follows from (5.2.12) that $Y(t)$ satisfies

$$\frac{dY}{dt} \le \frac{D(\mathcal{F})}{\sqrt{2}} - \Big[\frac{KNC^\infty}{D_0} - \Delta\mathcal{F}\Big]Y, \qquad t \in [0, T_*].$$

By Gronwall's lemma, we then have

$$\begin{aligned} Y(t) &\le \frac{D(\mathcal{F})/\sqrt{2}}{\frac{KNC^\infty}{D_0} - \Delta\mathcal{F}} \\ &\quad + \Big(Y(0) - \frac{D(\mathcal{F})/\sqrt{2}}{\frac{KNC^\infty}{D_0} - \Delta\mathcal{F}}\Big)\exp\Big[-\Big(\frac{KNC^\infty}{D_0} - \Delta\mathcal{F}\Big)t\Big]. \end{aligned}$$

This implies

$$Y(t) \le \max\Big\{Y(0), \frac{D(\mathcal{F})/\sqrt{2}}{\frac{KNC^\infty}{D_0} - \Delta\mathcal{F}}\Big\}. \tag{5.2.13}$$

On the other hand, note that the condition on K and the initial configuration are equivalent to saying that the r.h.s. of the above relation is less than or equal to $\frac{D_0}{\sqrt{2}}$.

$$\begin{aligned} &Y(0) = \sqrt{\mathcal{V}(\Theta_0)} < \frac{D_0}{\sqrt{2N}}, \quad \text{and} \\ &\frac{D(\mathcal{F})/\sqrt{2}}{\frac{KNC^\infty}{D_0} - \Delta\mathcal{F}} < \frac{D_0}{\sqrt{2N}} \quad \Longleftrightarrow \quad K > \frac{D(\mathcal{F}) + \frac{D_0}{\sqrt{N}}\Delta\mathcal{F}}{\sqrt{N}C^\infty}. \end{aligned} \tag{5.2.14}$$

Thus, we combine (5.2.13) and (5.2.14) to obtain

$$Y(t) < \frac{D_0}{\sqrt{2N}}, \quad \text{i.e.,} \quad \mathcal{V}(\Theta) < \frac{D_0^2}{2N}, \quad t \in [0, T_*).$$

This again yields for $t \in [0, T_*)$

$$\begin{aligned} |\theta_i(t) - \theta_j(t)| = |\hat{\theta}_i(t) - \hat{\theta}_j(t)| &\le \sqrt{2(|\hat{\theta}_i(t)|^2 + |\hat{\theta}_j(t)|^2)} \\ &\le \sqrt{2N\mathcal{V}(\Theta)} < D_0. \end{aligned} \tag{5.2.15}$$

Hence, we have

$$\lim_{t \to T_*-} D(\Theta(t)) < D_0,$$

which contradicts (5.2.11). Therefore $T_* = \infty$ and we obtain the desired uniform bound for $D(\Theta(t))$. $\square$

We are now ready to provide our second main theorem by combining the results of Lemmas 5.1.3 and 5.2.1.

The proof of Theorem 5.2.1. We repeat the argument presented in Lemma 5.2.1 to derive the estimate

$$\begin{aligned} Y(t) &= \sqrt{\mathcal{V}(\Theta)} \\ &\le \frac{D(\mathcal{F})/\sqrt{2}}{\frac{KNC^\infty}{D_0} - \Delta\mathcal{F}} + \Big(Y(0) - \frac{D(\mathcal{F})/\sqrt{2}}{\frac{KNC^\infty}{D_0} - \Delta\mathcal{F}}\Big) \exp\Big[-\Big(\frac{KNC^\infty}{D_0} - \Delta\mathcal{F}\Big)t\Big]. \end{aligned}$$

By letting $t \to \infty$, we obtain

$$\limsup_{t \to \infty} \sqrt{\mathcal{V}(\Theta)} \le \frac{D(\mathcal{F})/\sqrt{2}}{\frac{KNC^\infty}{D_0} - \Delta\mathcal{F}}.$$

On the other hand, from (5.2.15), note that

$$|\theta_i(t) - \theta_j(t)| \le \sqrt{2N\mathcal{V}(\Theta)}.$$

This implies

$$\limsup_{t \to \infty} |\theta_i(t) - \theta_j(t)| \le \limsup_{t \to \infty} \sqrt{2N\mathcal{V}(\Theta)} \le \frac{D(\mathcal{F})\sqrt{N}}{\frac{KNC^\infty}{D_0} - \Delta\mathcal{F}}$$

which leads to the desired result.

Remark 5.2.1. *1. Complete synchronization estimates for the Kuramoto model have been investigated in [16, 18, 25, 39, 45, 55].*

2. For the Kuramoto model with $F_i = \Omega_i$ and $\psi_{ij} = \frac{1}{N}$, we have

$$D(\mathcal{F}) = D(\Omega), \quad \Delta\mathcal{F} = 0, \quad C^\infty = \frac{\sin D_0}{N}.$$

Thus, the conditions on Ω_i, K and the initial configuration θ_0 in Lemma 5.2.1 reduce to

$$D(\Omega) < \infty, \quad K > \frac{D(\Omega)\sqrt{N}}{\sin D_0}, \quad D(\Theta_0) < D_0 < \pi, \quad \mathcal{V}(\Theta_0) < \frac{D_0^2}{2N},$$

which are weaker than in [16].

3. Note that for the linear stable dynamics

$$F_i(\theta_i, t) = a_i\theta_i, \ a_i \le 0, \tag{5.2.16}$$

we have

$$D(\mathcal{F}) = \infty, \quad \Delta\mathcal{F} = \max_i a_i < \infty.$$

Thus, Theorem 5.2.1 cannot be applied to this simple case where the decoupled system has bounded solutions only. However, if we check the proof of Lemma 5.2.1 more carefully, what we need is boundedness over the bounded phase space, not over the whole space $\mathbb{R}$ for θ_i, i.e., once the coupled system (5.0.3) *has only bounded solutions that are confined to the compact state space $\mathcal{N}$, then we can replace $D(\mathcal{F}) = \max_{i,j} \|F_i - F_j\|_{L^\infty(\mathbb{R}\times\mathbb{R}_+)}$ with a more relaxed diameter $\bar{D}(\mathcal{F}) := \max_{i,j} \|F_i - F_j\|_{L^\infty(\mathcal{N}\times\mathbb{R}_+)}$. With this relaxed definition for the diameter of $\mathcal{F}$, we can still use the result of Theorem 5.2.1 for* (5.2.16).

4. For a linear-time varying multi-agent systems, the practical synchronization has been studied in [47].

Below, we will show that if the uncoupled system (5.0.1) has a bounded solution for a given initial configuration, then the solution to the coupled system (5.0.3) exhibits practical synchronization.

Corollary 5.2.1. *Assume that the following conditions hold.*

1. *The initial configuration satisfies the following boundedness condition:*

$$D(\Theta_0) < D_0 < \pi, \qquad \mathcal{V}(\Theta_0) < \frac{D_0^2}{2N}.$$

2. *For a given family of forcing* $\mathcal{F} = \{F_1, \cdots, F_N\}$*, the decoupled system*

$$\dot{\theta}_i = F_i(\theta_i, t), \quad t > 0, \qquad \theta_i(0) = \theta_{i0}, \quad t = 0$$

   has the globally bounded solution:

$$\theta_i \in \mathcal{N}: \ \textit{compact subset of}\ \mathbb{R}.$$

3. *The family of forcing* $\mathcal{F}$*, network structure, and coupling strength satisfy the conditions:*

$$\bar{D}(\mathcal{F}) < \infty, \qquad \Delta\mathcal{F} < \infty, \qquad K > \frac{\bar{D}(\mathcal{F})\sqrt{N} + D_0}{NC^\infty}.$$

Then, the practical synchronization holds:

$$\lim_{K\to\infty} \lim_{t\to\infty} D(\Theta(t)) = 0.$$

Proof. It follows from Lemma 5.1.4 that the state space for Θ is bounded, so we can use the modified diameter $\bar{D}(\mathcal{F})$:

$$\bar{D}(\mathcal{F}) := \max_{1\le i,j\le N} \|F_i - F_j\|_{L^\infty(\mathcal{N}\times\mathbb{R}_+)}.$$

to apply the same argument as in Theorem 5.2.1. This completes the proof. □

Remark 5.2.2. *Corollary 5.2.1 covers the case where* F_i *is given by the gradient field of the double well potential, i.e.,*

$$F_i(\theta) = -\partial_\theta \varphi_i, \quad \varphi_i(\theta) = a_i\Big(\frac{\theta^2}{2} - \frac{\theta^4}{4}\Big), \quad a_i < 0.$$

In this case, the solution to the decoupled system is bounded.

5.2.2 Unbounded forcing

In this subsection, we consider the case where the decoupled system has bounded and unbounded solutions at the same time, and see that the unbounded solutions can be turned into bounded solutions by coupling with bounded solutions.

Consider a nonlinear system with linear intrinsic dynamics $\mathcal{F}_i(\theta, t) = p_i\theta$. Then the system (5.0.3) with all-to-all coupling $\psi_{ij} = \frac{1}{N}$ becomes

$$\begin{cases} \dot{\theta}_i = p_i\theta_i + \dfrac{K}{N}\displaystyle\sum_{j=1}^{N}\sin(\theta_j - \theta_i), & t > 0; \\ \theta_i(0) = \theta_{i0} & t = 0, \end{cases} \tag{5.2.17}$$

where p_i is a constant satisfying the negative sum condition:

$$\sum_{i=1}^{N} p_i < 0. \tag{5.2.18}$$

Note that system (5.2.17) has a trivial equilibrium solution Θ_e:

$$\Theta_e := (0, \cdots, 0).$$

When the coupling is turned off, i.e., $K = 0$, the state θ_i can go to infinity or zero exponentially fast, depending on the sign of p_i. If all p_i are negative, then the uncoupled dynamics have a bounded solution, and this case can be covered by Corollary 5.2.1. Thus without loss of generality, we may assume that at least one of the p_i is positive. In a near-equilibrium regime $\Theta \approx \Theta_e = 0$, the dynamics of the nonlinear system (5.2.17) can be studied via the linear system near Θ_e:

$$\dot{\theta}_i = p_i\theta_i + \frac{K}{N}\sum_{j=1}^{N}(\theta_j - \theta_i), \quad t > 0. \tag{5.2.19}$$

Before we present a uniform boundedness of Θ to the linear system (5.2.19) for sufficiently large K, we consider the following dynamics for two oscillators:

$$\begin{aligned} \dot{\theta}_1 &= p_1\theta_1 + \frac{K}{2}(\theta_2 - \theta_1), \quad t > 0, \\ \dot{\theta}_2 &= p_2\theta_2 + \frac{K}{2}(\theta_1 - \theta_2). \end{aligned} \tag{5.2.20}$$

The linear system (5.2.20) can be rewritten in matrix form as

$$\frac{d}{dt}\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} = M_2\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}, \qquad M_2 := \begin{bmatrix} p_1 - \frac{K}{2} & \frac{K}{2}\\ \frac{K}{2} & p_2 - \frac{K}{2}\end{bmatrix}.$$

By direct calculation, the matrix M_2 has two real eigenvalues:

$$\lambda_\pm := \frac{p_1 + p_2 - K \pm \sqrt{(p_1 - p_2)^2 + K^2}}{2}.$$

It is easy to see that $\lambda_- < 0$ for sufficiently large K. On the other hand, note that

$$\lambda_+ < 0 \qquad \Longleftrightarrow \qquad p_1 + p_2 < 0, \quad K > \frac{2p_1p_2}{p_1+p_2} > 0.$$

Thus, if $p_1 + p_2 < 0, \quad K > \frac{2p_1p_2}{p_1+p_2}$, then both θ_1 and θ_2 decay to zero so that we have practical synchronization. Before we proceed to the general case, we consider the two explicit examples corresponding to the case $p_1 + p_2 > 0$. In this case, we will not have practical synchronization.

• **Example 1** $(p_1, p_2) = (1, 2)$**.** In this case, system (5.2.20) becomes

$$\begin{aligned}\dot{\theta}_1 &= \theta_1 + \frac{K}{2}(\theta_2 - \theta_1), \quad t > 0,\\ \dot{\theta}_2 &= 2\theta_2 + \frac{K}{2}(\theta_1 - \theta_2).\end{aligned}$$

By direct calculation, the solution (θ_1, θ_2) satisfies

$$|\theta_2(t) - \theta_1(t)| = Ce^{\lambda_+(K)t} \to \infty, \quad t \to \infty.$$

Thus, we do not have practical synchronization.

• **Example 2** $(p_1, p_2) = (-1, 2)$**.**

$$\begin{aligned}\dot{\theta}_1 &= -\theta_1 + \frac{K}{2}(\theta_2 - \theta_1), \quad t > 0,\\ \dot{\theta}_2 &= 2\theta_2 + \frac{K}{2}(\theta_1 - \theta_2).\end{aligned}$$

Again, by direct calculation, we have

$$|\theta_2(t) - \theta_1(t)| = Ce^{\lambda_+(K)t} \to \infty, \quad t \to \infty.$$

Thus, we can conclude that with $p_1 + p_2 > 0$, system (5.2.20) cannot have a practical synchronization.

We now return to the linear system (5.2.19) associated with (5.2.17) rewritten in matrix form:

$$\dot{\theta} = M_N \theta, \quad t > 0, \qquad \theta = (\theta_1, \cdots, \theta_N), \tag{5.2.21}$$

where

$$M_N := \begin{pmatrix} p_1 - \frac{N-1}{N}K & \frac{K}{N} & \cdots & \frac{K}{N} \\ \frac{K}{N} & p_2 - \frac{N-1}{N}K & \cdots & \frac{K}{N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{K}{N} & \frac{K}{N} & \cdots & p_N - \frac{N-1}{N}K \end{pmatrix} \tag{5.2.22}$$

Note that the coefficient matrix M_N is symmetric, so that all eigenvalues of M_N are real. Below, we will show that, under the condition (5.2.18), if the coupling strength K is sufficiently large, then all eigenvalues of M_N become negative so that the trivial equilibrium solution to (5.2.21) is exponentially stable. For this, we recall several lemmas in relation to eigenvalues of this linearized system.

Lemma 5.2.2 (Weyl's inequality [75])**.** *Let M, H, and P be Hermitian matrices satisfying $M = H + P$, and suppose $\{\mu_1, \ldots, \mu_N\}, \{\nu_1, \ldots, \nu_N\}$ and $\{\rho_1, \ldots, \rho_N\}$ are (as known) real eigenvalues of M, H, and P, respectively, such that*

$$\mu_1 \geq \cdots \geq \mu_N, \quad \nu_1 \geq \cdots \geq \nu_N, \quad \rho_1 \geq \cdots \geq \rho_N.$$

Then, the following inequalities hold:

$$\mu_j \leq \nu_i + \rho_{j-i+1}, \qquad 1 \leq i, j \leq N \quad \text{and} \quad i \leq j.$$

Proof. For a proof, we refer to Theorem III.2.1 in [8]. □

Lemma 5.2.3. *The matrix* (5.2.22) *has a determinant of the form*

$$\det M_N = \frac{(-1)^{N-1}}{N} \Big(\sum_{i=1}^{N} p_i \Big) K^{N-1} + \mathcal{O}(K^{N-2}) \quad \text{as a polynomial in } K.$$

Proof. The proof is given in Appendix B. $\square$

Now, we are ready to prove the negativity of eigenvalues of M_N. For a matrix A, let $\sigma(A)$ denote the set of eigenvalues of A, i.e., the spectrum of A.

Proposition 5.2.1. *Let $\Theta = (\theta_1, \cdots, \theta_N)$ be a global solution to system* (5.2.19) *with negative sum condition* (5.2.18)*. Then, for a sufficiently large K, the solution Θ converges to zero exponentially fast, independent of the initial configuration Θ_0.*

Proof. For the desired estimate, it suffices to show that the eigenvalues for the coefficient matrix M_N are negative:

$$0 > \mu_1 > \mu_2 > \cdots > \mu_N.$$

Without loss of generality, we may assume that $p_1 \geq p_2 \geq \ldots \geq p_N$. Suppose that $\sigma(M_N) = \{\mu_1, \ldots, \mu_N\}$ is arranged in descending order, and set H_N and P_N as

$$H_N := \begin{pmatrix} \frac{K}{N} & \frac{K}{N} & \ldots & \frac{K}{N} \\ \frac{K}{N} & \frac{K}{N} & \ldots & \frac{K}{N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{K}{N} & \frac{K}{N} & \ldots & \frac{K}{N} \end{pmatrix}, \qquad P_N := \begin{pmatrix} p_1 - K & 0 & \ldots & 0 \\ 0 & p_2 - K & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & p_N - K \end{pmatrix}$$

so that

$$M_N = H_N + P_N.$$

Then, it is easy to see that the spectra of the matrices H_N and P_N are given in descending order:

$$\sigma(H_N) = \{K, 0, \ldots, 0\}, \qquad \sigma(P_N) = \{p_1 - K, p_2 - K, \ldots, p_N - K\}.$$

It follows from Lemma 5.2.2 that we have

$$\mu_2 \leq K + (p_2 - K) = p_2 \quad \text{and} \quad \mu_2 \leq 0 + (p_1 - K) = p_1 - K.$$

We now set K to be sufficiently large satisfying

$$K > p_1. \tag{5.2.23}$$

Therefore, under the condition (5.2.23), μ_2 is negative, and hence $\mu_i,\ i = 3, \cdots, N$ are also negative:

$$0 > \mu_2 \geq \mu_3 \geq \cdots \geq \mu_N. \tag{5.2.24}$$

Now, we need to show that μ_1 is negative. Since the determinant of a matrix is the product of all eigenvalues, it is enough to show that

$$\mu_1 < 0 \quad \text{if and only if} \quad \det M_N \begin{cases} < 0, & \text{if } N \text{ is odd,} \\ > 0, & \text{if } N \text{ is even.} \end{cases}$$

In Lemma 5.2.3, for sufficiently large K,

$$\begin{aligned} \det M_N &= \mu_1 \mu_2 \cdots \mu_N \\ &= \mu_1 (-1)^{N-1} |\mu_2| \cdots |\mu_N| \\ &\approx \frac{(-1)^{N-1}}{N} \Big(\sum_{i=1}^N p_i \Big) K^{N-1} = \frac{(-1)^N}{N} \Big| \sum_{i=1}^N p_i \Big| K^{N-1}. \end{aligned}$$

This yields that, for sufficiently large K,

$$\mu_1 < 0. \tag{5.2.25}$$

It follows from (5.2.24)-(5.2.25) that all eigenvalues of M_N are negative for sufficiently large K. This implies that the solution to the linear system (5.2.19) decays to zero exponentially fast. $\square$

Chapter 6

Interplay of inertia and heterogeneous dynamics

In this chapter, we study the effects of inertia as in [3, 18, 74, 76], on the Kuramoto system with heterogeneous intrinsic dynamics which is studied in the previous Chapter 5. We use the notation $\theta_i \in \mathbb{R}$ to denote the i-th node oscillator, and consider the uniform all-to-all network structure for the Kuramoto sysetm with inertia m:

$$m\ddot{\theta}_i + \dot{\theta}_i = F_i(\theta_i, t) + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad i = 1, \cdots, N, \tag{6.0.1}$$

where the forcing $F_i(\theta_i, t)$ is Lipschitz continuous. The contents of this chapter is based on the joint work in [43]

6.1 Heterogenous Kuramoto oscillators

We first introduce an averaged variable (macro variable), and a fluctuation around this (micro variable): for $\Theta = (\theta_1, \cdots, \theta_N) \in \mathbb{R}^N$, we set

$$\begin{aligned}
&\omega_i = \dot{\theta}_i, \quad \theta_c := \frac{1}{N}\sum_{i=1}^{N}\theta_i, \quad \omega_c := \frac{1}{N}\sum_{i=1}^{N}\omega_i, \\
&\hat{\theta}_i := \theta_i - \theta_c, \quad \hat{\omega}_i := \omega_i - \omega_c, \quad 1 \le i \le N.
\end{aligned}$$

We next present some notations that will used throught out this chapter.
Notation: For $\zeta := (\zeta_1, \ldots, \zeta_N), \xi := (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$ and $\mathcal{F} := \{F_i \mid F_i : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R},\ 1 \le i \le N\}$, we set

$$
\begin{aligned}
&\langle \zeta, \xi \rangle := \sum_{i=1}^N \zeta_i \xi_i, \quad \|\zeta\| := \Big(\sum_{i=1}^N |\zeta_i|^2 \Big)^{\frac{1}{2}}, \quad \zeta_m := \min_{1 \le i \le N} \zeta_i, \quad \zeta_M := \max_{1 \le i \le N} \zeta_i, \\
&D(\zeta) := \zeta_M - \zeta_m, \quad D(\dot{\zeta}) := \max_{i,j} |\dot{\zeta}_i - \dot{\zeta}_j|, \quad \Delta \mathcal{F} := \sup_{i,\zeta,t} \frac{\partial F_i}{\partial \zeta}(\zeta, t), \\
&D(\mathcal{F}) := \max_{1 \le i,j \le N} \|F_i - F_j\|_{L^\infty(\mathbb{R} \times \mathbb{R}_+)}, \\
&B_L(\mathcal{F}) := \sup_{t \ge 0} \max_{1 \le i \le N} \sup \Big\{ \frac{|F_i(x) - F_i(y)|^2}{|x-y|^2} \ \Big| \ x, y \in \mathbb{R},\ x \ne y \Big\}.
\end{aligned}
$$

Throughout this chapter, C denotes some generic positive constant, which may take different values in different places. Furthermore, $A \lesssim B$ means that there is a generic constant $C > 0$ such that $A \le CB$, while $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$.

We consider heterogeneous forcing functions $\mathcal{F} = \{F_1, \cdots, F_N\}$:

$$
\exists\ i \ne j \quad \text{such that} \quad F_i \not\equiv F_j.
$$

6.1.1 Energy functional and main result

We study a decay estimate of the Lyapunov functional, using an energy functional approach. For this, we introduce an energy functional:

$$
\begin{aligned}
&\mathcal{E}(t) := \mathcal{E}_k(t) + \mathcal{E}_p(t), \\
&\mathcal{E}_k(t) := m\|\dot{\Theta}\|^2 + \frac{2}{3}\langle \dot{\Theta}, \Theta \rangle + \frac{1}{3m}\|\Theta\|^2, \quad \mathcal{E}_p(t) := \frac{K}{N} \sum_{i,j=1}^N (1 - \cos(\theta_j - \theta_i)).
\end{aligned}
$$

In the next Lemma, we will show that the functional $\mathcal{E}$ is equivalent to $K\|\Theta\|^2 + \|\dot{\Theta}\|^2$.

Lemma 6.1.1. *Let $\Theta = \Theta(t)$ be a solution to* (6.0.1)*, satisfying the a priori condition: For a positive constant T,*

$$
\max_{t \in [0,T)} D(\Theta(t)) < D_\infty < \pi.
$$

Then, the functional $\mathcal{E}(t)$ and $K\|\Theta\|^2 + \|\dot{\Theta}\|^2$ are equivalent in the sense that there exist positive constants C_0 and C_1 depending only on $m, K,$ and D_∞, such that

$$C_0(K\|\Theta\|^2 + \|\dot{\Theta}\|^2) \le \mathcal{E}(t) \le C_1(K\|\Theta\|^2 + \|\dot{\Theta}\|^2), \quad t \in [0, T).$$

Proof. Below, we estimate $\mathcal{E}_k$ and $\mathcal{E}_p$ separately.

• (Estimate of $\mathcal{E}_k$): We use the Cauchy-Schwarz inequality and Young's inequality to find that

$$|\langle \dot{\Theta}, \Theta \rangle| \le |\dot{\Theta}||\Theta| \le \frac{3m}{4}\|\dot{\Theta}\|^2 + \frac{1}{3m}\|\Theta\|^2,$$

Then, the above relation yields

$$\frac{m}{2}\|\dot{\Theta}\|^2 \le \mathcal{E}_k(t) \le \frac{3m}{2}\|\dot{\Theta}\|^2 + \frac{5}{9m}\|\Theta\|^2 \tag{6.1.1}$$

• (Estimate of $\mathcal{E}_p$): Through a straightforward calculation, we have

$$\frac{1 - \cos D_\infty}{D_\infty^2}|\theta_j - \theta_i|^2 \le 1 - \cos(\theta_j - \theta_i) \le \frac{1}{2}|\theta_j - \theta_i|^2, \qquad |\theta_j - \theta_i| \le D_\infty < \pi.$$

Since $\displaystyle\sum_{i,j=1}^{N} |\theta_j - \theta_i|^2 = 2N\|\Theta\|^2$, we have

$$2K\Big(\frac{1 - \cos D_\infty}{D_\infty^2}\Big)\|\Theta\|^2 \le \mathcal{E}_p(t) \le K\|\Theta\|^2. \tag{6.1.2}$$

Finally, we combine (6.1.1) and (6.1.2), to obtain

$$\begin{aligned} \mathcal{E}(t) &\ge 2\Big(\frac{1 - \cos D_\infty}{D_\infty^2}\Big)K\|\Theta\|^2 + \frac{m}{2}\|\dot{\Theta}\|^2, \\ \mathcal{E}(t) &\le \Big[\frac{5}{9mK} + 1\Big]K\|\Theta\|^2 + \frac{3m}{2}\|\dot{\Theta}\|^2. \end{aligned}$$

and set C_0 and C_1 as follows:

$$C_0 := \min\left\{\frac{m}{2}, \frac{2(1 - \cos D_\infty)}{D_\infty^2}\right\}, \qquad C_1 := \max\left\{\frac{3m}{2}, 1 + \frac{5}{9mK}\right\}$$

With this, we obtain the desired result. $\square$

In this setting, we have the following asymptotic practical synchronization.

Theorem 6.1.1. *There exists K_∞, such that if $K > K_\infty$ and the initial data $\big(\Theta(0), \dot{\Theta}(0)\big) = (\phi, \psi)$ satisfy the relations*

$$D(\phi) < D_\infty < \pi \qquad \text{and} \qquad \mathcal{E}(0) < \frac{C_0 K D_\infty^2}{2},$$

then, for any solution $\Theta = \Theta(t)$ with initial data (ϕ, ψ), we have the practical synchronization:

$$\lim_{K\to\infty} \limsup_{t\to\infty} D(\Theta(t)) = 0,$$

where the positive constant C_0 is defined by the following relation:

$$C_0 := \min\left\{\frac{m}{2}, \frac{2(1-\cos D_\infty)}{D_\infty^2}\right\}$$

Remark 6.1.1. *1. In the proof of theorem 6.1.1, we will show that the following condition holds for a sufficiently large K:*

$$\limsup_{t\to\infty} \sqrt{K\|\Theta(t)\|^2 + \|\dot{\Theta}(t)\|^2} \le C(m, D_\infty, \Delta\mathcal{F}, B_L(\mathcal{F})).$$

We use this condition to obtain the practical synchronization.

2. For the Kuramoto model with $F_i(\Theta) = \Omega_i$, we have

$$D(\mathcal{F}) = D(\Omega), \quad \Delta\mathcal{F} = 0, \quad B_L(\mathcal{F}) = 0.$$

With this, the condition on $\mathcal{F}$, K, and the initial data $(\phi,\, \psi)$ is reduced to

$$\begin{aligned} & D(\Omega) < \infty, \quad D(\phi) \le D_\infty < \pi, \\ & K > \frac{4\sqrt{2N} D(\Omega) \max\left\{\frac{3}{2m}, 1 + \frac{5}{9m}\right\} \times \max\left\{\frac{1}{3m}, 1\right\}}{\left(\min\left\{\frac{m}{2}, \frac{2(1-\cos D_\infty)}{D_\infty^2}\right\}\right)^{\frac{1}{2}} \times \min\left\{\frac{1}{3m}, \frac{2\sin D_\infty}{3mD_\infty}\right\}} \end{aligned}$$

6.1.2 Energy estimates

Consider the dynamics for (θ_i, ω_i):

$$\begin{cases} \dot{\theta}_i = \omega_i, \quad t > 0, \\ \dot{\omega}_i = \dfrac{1}{m}\Big[-\omega_i + \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \big(F_i(\theta_c + \theta_i, t) - F_j(\theta_c + \theta_j, t)\big) + \dfrac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i)\Big], \\ \displaystyle\sum_{i=1}^{N} \theta_i = 0 \quad \text{and} \quad \sum_{i=1}^{N} \omega_i = 0. \end{cases} \tag{6.1.3}$$

Lemma 6.1.2. *For $T \in (0, \infty]$, let θ be a solution to* (6.0.1)*, satisfying the a priori condition*

$$\max_{t\in[0,T)} D(\Theta(t)) \le D_\infty < \pi. \tag{6.1.4}$$

Then, we have

$$(i)\ \frac{d}{dt}\Big[m\|\dot{\Theta}\|^2 + \frac{K}{N}\sum_{i,j=1}^{N}(1 - \cos(\theta_j - \theta_i))\Big] \le -\|\dot{\Theta}\|^2 + NB_L(\mathcal{F})\|\Theta\|^2 + 2\sqrt{N}D(\mathcal{F})\|\dot{\Theta}\|.$$

$$(ii)\ \frac{d}{dt}\Big[2m\langle\dot{\Theta}, \Theta\rangle + \|\Theta\|^2\Big] \le \Big(-2KR_\infty + 2\Delta\mathcal{F}\Big)\|\Theta\|^2 + 2m\|\dot{\Theta}\|^2 + 2\sqrt{N}D(\mathcal{F})\|\Theta\|.$$

Proof. (i) For the estimate of the first assertion, we multiply both sides of the second equation in (6.1.3) by $2\dot{\theta}_i$ and sum over i, to obtain

$$\begin{aligned} m\frac{d}{dt}\|\dot{\Theta}\|^2 &= -2\|\dot{\Theta}\|^2 + \frac{2}{N}\sum_{i,j=1}^{N}\big(F_i(\theta_c + \theta_i, t) - F_j(\theta_c + \theta_j, t)\big)\dot{\theta}_i \\ &\quad + \frac{2K}{N}\sum_{i,j=1}^{N}\sin(\theta_j - \theta_i)\dot{\theta}_i \\ &= -2\|\dot{\Theta}\|^2 + \frac{1}{N}\sum_{i,j=1}^{N}\big(F_i(\theta_c + \theta_i, t) - F_j(\theta_c + \theta_j, t)\big)(\dot{\theta}_i - \dot{\theta}_j) \end{aligned}$$

$$
\begin{aligned}
&- \frac{K}{N} \sum_{i,j=1}^{N} (\dot{\theta}_j - \dot{\theta}_i) \sin(\theta_j - \theta_i) \\
&=: -2\|\dot{\Theta}\|^2 + \mathcal{I}_{31} + \mathcal{I}_{32}.
\end{aligned} \tag{6.1.5}
$$

In the following, we estimate the terms $\mathcal{I}_{3i}$ separately.

• Case A (Estimate of $\mathcal{I}_{31}$): In this case, we use the relation

$$
F_i(\theta_c+\theta_i,t)-F_j(\theta_c+\theta_j,t) = F_i(\theta_c+\theta_i,t)-F_j(\theta_c+\theta_i,t)+F_j(\theta_c+\theta_i,t)-F_j(\theta_c+\theta_j,t)
$$

to find

$$
\begin{aligned}
|\mathcal{I}_{31}| &\le \frac{1}{N} \sum_{i,j=1}^{N} \Big| \big(F_i(\theta_c+\theta_i,t) - F_j(\theta_c+\theta_i,t)\big)(\dot{\theta}_i - \dot{\theta}_j) \Big| \\
&+ \frac{1}{N} \sum_{i,j=1}^{N} \Big| \big(F_j(\theta_c+\theta_i,t) - F_j(\theta_c+\theta_j,t)\big)(\dot{\theta}_i - \dot{\theta}_j) \Big| \\
&=: \mathcal{I}_{311} + \mathcal{I}_{312}.
\end{aligned} \tag{6.1.6}
$$

◇ (Estimate of $\mathcal{I}_{311}$): In this case, we have

$$
\mathcal{I}_{311} \le 2D(\mathcal{F}) \sum_{i=1}^{N} |\dot{\theta}_i| \le 2\sqrt{N} D(\mathcal{F}) \|\dot{\Theta}\|. \tag{6.1.7}
$$

◇ (Estimate of $\mathcal{I}_{312}$): We use the Lipschitz continuity of F_j, to derive

$$
\begin{aligned}
&\big| \big(F_j(\theta_c+\theta_i,t) - F_j(\theta_c+\theta_j,t)\big)(\dot{\theta}_i - \dot{\theta}_j) \big| \\
&\le \sup_{t\ge 0} \|F_j(\cdot,t)\|_{Lip} |\theta_i - \theta_j| |\dot{\theta}_i - \dot{\theta}_j| \le \frac{|\dot{\theta}_i - \dot{\theta}_j|^2}{2N} + \frac{N}{2} B_L(\mathcal{F}) |\theta_i - \theta_j|^2.
\end{aligned}
$$

Then, we can obtain that

$$
\mathcal{I}_{312} \le \|\dot{\Theta}\|^2 + N B_L(\mathcal{F}) \|\Theta\|^2. \tag{6.1.8}
$$

In (6.1.6), we combine (6.1.7) and (6.1.8), to obtain

$$
|\mathcal{I}_{31}| \le 2\sqrt{N} D(\mathcal{F}) \|\dot{\Theta}\| + \|\dot{\Theta}\|^2 + N B_L(\mathcal{F}) \|\Theta\|^2. \tag{6.1.9}
$$

$\diamond$ Case B (Estimate of $\mathcal{I}_{32}$): Note that the term $\mathcal{I}_{32}$ can be rewritten as follows:

$$\begin{aligned}\mathcal{I}_{32} &= -\frac{K}{N}\sum_{i,j=1}^{N}(\dot\theta_j-\dot\theta_i)\sin(\theta_j-\theta_i) = \frac{d}{dt}\Big(\frac{K}{N}\sum_{i,j=1}^{N}\cos(\theta_j-\theta_i)\Big)\\ &= \frac{K}{N}\frac{d}{dt}\sum_{i,j=1}^{N}\Big(-1+\cos(\theta_j-\theta_i)\Big).\end{aligned} \tag{6.1.10}$$

Finally, in (6.1.5), we combine (6.1.9) and (6.1.10) to obtain the desired estimate.

(ii) We now multiply both sides of (6.1.3) by $2\theta_i$, take a sum over i, and use the symmetric property for $i \leftrightarrow j$, to obtain

$$\begin{aligned}2m\sum_{i=1}^{N}\ddot\theta_i\theta_i &= -\frac{d}{dt}\sum_{i=1}^{N}\theta_i^2 + \frac{2}{N}\sum_{i,j=1}^{N}\Big(F_i(\theta_c+\theta_i,t)-F_j(\theta_c+\theta_j,t)\Big)\theta_i\\ &\quad+\frac{2K}{N}\sum_{i,j=1}^{N}\theta_i\sin(\theta_j-\theta_i)\\ &= -\frac{d}{dt}\|\Theta\|^2 + \frac{1}{N}\sum_{i,j=1}^{N}\Big(F_i(\theta_c+\theta_i,t)-F_j(\theta_c+\theta_j,t)\Big)(\theta_i-\theta_j)\\ &\quad-\frac{K}{N}\sum_{i,j=1}^{N}(\theta_j-\theta_i)\sin(\theta_j-\theta_i)\\ &=: -\frac{d}{dt}\|\Theta\|^2 + \mathcal{I}_{41}+\mathcal{I}_{42}.\end{aligned} \tag{6.1.11}$$

$\bullet$ (Estimate of L.H.S. of (6.1.11)): Note that the L.H.S. of (6.1.11) can be rewritten as follows:

$$2m\sum_{i=1}^{N}\ddot\theta_i\theta_i = \frac{d}{dt}\sum_{i=1}^{N}(2m\dot\theta_i\theta_i) - 2m\sum_{i=1}^{N}|\dot\theta_i|^2 = 2m\frac{d}{dt}\langle\dot\Theta,\Theta\rangle - 2m\|\dot\Theta\|^2. \tag{6.1.12}$$

• (Estimate of R.H.S. of (6.1.11)): The term $\mathcal{I}_{41}$ can be treated using the same argument, as follows:

$$
\begin{aligned}
\mathcal{I}_{41} &\le \frac{1}{N}\sum_{i,j=1}^{N}\big(F_i(\theta_c+\theta_i,t)-F_j(\theta_c+\theta_j,t)\big)(\theta_i-\theta_j)\\
&\le \frac{1}{N}\sum_{i,j=1}^{N}\big(F_i(\theta_c+\theta_i,t)-F_j(\theta_c+\theta_i,t)\big)(\theta_i-\theta_j)\\
&+\frac{1}{N}\sum_{i,j=1}^{N}\big(F_j(\theta_c+\theta_i,t)-F_j(\theta_c+\theta_j,t)\big)(\theta_i-\theta_j)\\
&\le \frac{D(\mathcal{F})}{N}\sum_{i,j=1}^{N}(|\theta_i|+|\theta_j|)+\frac{\Delta\mathcal{F}}{N}\sum_{i,j=1}^{N}|\theta_i-\theta_j|^2\\
&\le 2\sqrt{N}D(\mathcal{F})\|\Theta\|+2\Delta\mathcal{F}\|\Theta\|^2.
\end{aligned}
\tag{6.1.13}
$$

For the estimate of $\mathcal{I}_{42}$, we use the assumption (6.1.4) to see that

$$
(\theta_j-\theta_i)\sin(\theta_j-\theta_i)\ge R_\infty|\theta_j-\theta_i|^2
$$

$$
\begin{aligned}
\mathcal{I}_{42} &= -\frac{K}{N}\sum_{i,j=1}^{N}(\theta_j-\theta_i)\sin(\theta_j-\theta_i)\\
&\le -\frac{KR_\infty}{N}\sum_{i,j=1}^{N}|\theta_j-\theta_i|^2=-2KR_\infty\|\Theta\|^2.
\end{aligned}
\tag{6.1.14}
$$

In (6.1.11), we use estimates the (6.1.12), (6.1.13), and (6.1.14) to obtain the desired estimate.

$$
\frac{d}{dt}\Big[2m\langle\dot{\Theta},\Theta\rangle+\|\Theta\|^2\Big]\le\Big(-2KR_\infty+2\Delta\mathcal{F}\Big)\|\Theta\|^2+2m\|\dot{\Theta}\|^2+2\sqrt{N}D(\mathcal{F})\|\Theta\|.
$$

□

Finally, we are ready to derive a differential inequality for $\mathcal{E}(t)$, as follows.

Proposition 6.1.1. *Suppose that the inertia strength m is positive. Then, there exists a coupling strength $K_\infty \geq 1$ such that if $K \geq K_\infty$, then for any solution $\Theta = \Theta(t)$ to (6.1.3) satisfying the a priori condition*

$$\max_{t\in[0,T)} D(\Theta) \leq D_\infty < \pi,$$

for some $T \in (0, \infty]$, there exist positive constants α and β such that

$$\frac{d}{dt}\mathcal{E}(t) \leq -\alpha\mathcal{E}(t) + \beta\sqrt{\mathcal{E}(t)}, \quad t \in [0, T).$$

Proof. It follows from Lemma 6.1.2 that we have

$$\begin{aligned}
&\frac{d}{dt}\Big[m\|\dot{\Theta}\|^2 + \frac{2}{3}\langle\dot{\Theta}, \Theta\rangle + \frac{1}{3m}\|\Theta\|^2 + \frac{K}{N}\sum_{i,j=1}^{N}(1-\cos(\theta_j - \theta_i))\Big] \\
&\quad \leq -\frac{1}{3}\|\dot{\Theta}\|^2 - \Big[\frac{2}{3m}R_\infty - \frac{\frac{2}{3m}\Delta\mathcal{F} + NB_L(\mathcal{F})}{K}\Big]K\|\Theta\|^2 \\
&\quad + 2\sqrt{N}D(\mathcal{F})(\frac{1}{3m}\|\Theta\| + \|\dot{\Theta}\|).
\end{aligned} \tag{6.1.15}$$

We set K_∞, to satisfy

$$\frac{2}{3m}R_\infty - \frac{\frac{2}{3m}\Delta\mathcal{F} + NB_L(\mathcal{F})}{K_\infty} > 0.$$

Then, the estimate (6.1.15) yields

$$\frac{d\mathcal{E}(t)}{dt} \leq -\bar{\alpha}(K\|\Theta\|^2 + \|\dot{\Theta}\|^2) + 2\sqrt{N}D(\mathcal{F})\max\Big\{\frac{1}{3m}, 1\Big\}(\|\Theta\| + \|\dot{\Theta}\|), \tag{6.1.16}$$

where the positive constant α is defined by the following relation:

$$\bar{\alpha} := \min\Big\{\frac{1}{3}, \frac{2R_\infty}{3m} - \frac{\frac{2\Delta\mathcal{F}}{3m} + NB_L(\mathcal{F})}{K_\infty}\Big\}.$$

On the other hand, it follows from Lemma 6.1.1 that we have

$$\begin{aligned}
&K\|\Theta\|^2 + \|\dot{\Theta}\|^2 \geq \frac{1}{C_1}\mathcal{E}(t), \quad \text{and} \\
&\|\Theta\| + \|\dot{\Theta}\| \leq \sqrt{2}\sqrt{\|\Theta\|^2 + \|\dot{\Theta}\|^2} \leq \sqrt{2}\sqrt{K\|\Theta\|^2 + \|\dot{\Theta}\|^2} \\
&\qquad \leq \sqrt{\frac{2}{C_0}}\sqrt{\mathcal{E}(t)}.
\end{aligned} \tag{6.1.17}$$

In (6.1.16), we use (6.1.17) to obtain

$$\frac{d}{dt}\mathcal{E}(t) \le -\frac{\bar{\alpha}}{C_1}\mathcal{E}(t) + 2\sqrt{\frac{2N}{C_0}}\max\left\{\frac{1}{3m}, 1\right\}D(\mathcal{F})\sqrt{\mathcal{E}(t)}.$$

We set

$$\alpha := \frac{\bar{\alpha}}{C_1}, \quad \beta := 2\sqrt{\frac{2N}{C_0}}\max\left\{\frac{1}{3m}, 1\right\}D(\mathcal{F}) \tag{6.1.18}$$

to obtain the desired result. □

Remark 6.1.2. *Note that the relation* (6.1.18) *implies that for $K \gg 1$,*

$$\frac{\beta}{\alpha} \lesssim \frac{\sqrt{N}}{\sqrt{C_0}}C_1 D(\mathcal{F}).$$

Recall that

$$\begin{aligned} C_0 &= \min\left\{\frac{m}{2}, \frac{2(1-\cos D_\infty)}{D_\infty^2}\right\}, \\ C_1 &= \max\left\{\frac{3}{2m}, 1 + \frac{5}{9mK}\right\} \le \max\left\{\frac{3}{2m}, 1 + \frac{5}{9m}\right\}, \end{aligned}$$

so we can obtain a bound on $\frac{\beta}{\alpha}$ independently of K:

$$\frac{\beta}{\alpha} \lesssim \sqrt{N}C(m, D_\infty)D(\mathcal{F}),$$

where $C(m, D_\infty)$ is a positive constant depending only on the inertia m and the phase diameter D_∞. We can observe that

$$C(m, D_\infty) = \frac{\max\left\{\frac{3}{2m}, 1 + \frac{5}{9m}\right\} \times \max\left\{\frac{1}{3m}, 1\right\}}{\min\left\{\frac{m}{2}, \frac{2(1-\cos D_\infty)}{D_\infty^2}\right\}^{\frac{1}{2}} \times \min\left\{\frac{1}{3}, \frac{2R_\infty}{3m}\right\}} \to \infty \quad \textit{as} \quad m \to 0.$$

6.1.3 The proof of Theorem 6.1.1

In this subsection, we present the proof of Theorem 6.1.1. It follows from Proposition 6.1.1 that we have

$$\frac{d}{dt}\mathcal{E}(t) \le -\alpha\mathcal{E}(t) + \beta\sqrt{\mathcal{E}(t)}, \quad t \in [0, T). \tag{6.1.19}$$

We split the proof into two parts.

• Step A (The Gronwall inequality (6.1.19) holds for $T = \infty$): For this, we set

$$Z(t) := \sqrt{\mathcal{E}(t)}, \quad t \geq 0,$$

Then, (6.1.19) can be rewritten as

$$\frac{dZ}{dt} \leq -\frac{\alpha}{2}Z + \frac{\beta}{2}.$$

Note the following relation:

$$|\theta_i(t) - \theta_j(t)|^2 \leq 2\|\Theta(t)\|_2^2 \leq \frac{2}{C_0 K}\mathcal{E}(t) = \frac{2}{C_0 K}Z(t)^2 < D_\infty^2. \tag{6.1.20}$$

By the assumption, we have

$$\mathcal{E}(0) < \frac{C_0 K D_\infty^2}{2}. \tag{6.1.21}$$

We define a set $\mathcal{T}$ and its supremum T^* as follows:

$$\mathcal{T} := \left\{ T \in \mathbb{R}_+ : Z(t) < \sqrt{\frac{C_0 K}{2}} D_\infty, \qquad \forall\, t \in [0, T) \right\}, \qquad T^* := \sup \mathcal{T}.$$

Note that the assumption (6.1.21) implies that

$$Z(0) < \sqrt{\frac{C_0 K}{2}} D_\infty,$$

and by continuity, there exists a positive constant $T_1 > 0$ such that $T_1 \in \mathcal{T}$, and

$$\frac{dZ}{dt} \leq -\frac{\alpha}{2}Z + \frac{\beta}{2}, \quad t \in [0, T^*). \tag{6.1.22}$$

We now claim that

$$T^* = \infty. \tag{6.1.23}$$

Suppose that this is not true, i.e., T^* is finite. Then, we should have that

$$Z(T^*) = \sqrt{\frac{C_0 K}{2}} D_\infty. \tag{6.1.24}$$

It follows from (6.1.22) that we have

$$Z(t) \le Z(0)e^{-\frac{\alpha}{2}t} + \frac{\beta}{\alpha}(1 - e^{-\frac{\alpha}{2}t}), \qquad t \in [0, T^*].$$

Letting $t \to T^*$, we have

$$\begin{aligned} Z(T^*) &\le Z(0)e^{-\frac{\alpha}{2}T^*} + \frac{\beta}{\alpha}(1 - e^{-\frac{\alpha}{2}T^*}) \\ &= \frac{\beta}{\alpha} + \Big(Z(0) - \frac{\beta}{\alpha}\Big)e^{-\frac{\alpha}{2}T^*} \\ &< \frac{\beta}{\alpha} + \Big(Z(0) - \frac{\beta}{\alpha}\Big) = Z(0) \\ &< \sqrt{\frac{C_0 K}{2}} D_\infty, \end{aligned}$$

where we use the fact that $K \gg 1$ to see that

$$Z(0) - \frac{\beta}{\alpha} > 0.$$

This contradicts (6.1.24). This completes the proof of the claim (6.1.23).

• Step B (Emergence of practical synchronization): It follows from (6.1.20) that

$$D(\Theta(t)) \le \sqrt{\frac{2}{C_0 K}} Z(t).$$

Letting $t \to \infty$, we obtain

$$\limsup_{t\to\infty} D(\Theta(t)) \lesssim \sqrt{\frac{2}{C_0 K}} \limsup_{t\to\infty} Z(t) \lesssim \frac{1}{\sqrt{K}}.$$

Thus, we have shown a practical synchronization.

Chapter 7

Kuramoto model with adative coupling

In the classical Kuramoto model (1.0.1), the coupling strength K is given by a constant. However, the large fixed coupling strength may not be necessary for the synchronization of oscillators which are already close to the synchronized states. Thus, it is plausible to consider the dynamics with varying coupling magnitude depending on the distance between oscillators. This adaptive coupling was recently addressed in [21, 62, 65] for biological and physical contexts. In this chapter, we study a self-consistent Kuramoto type model proposed in [1, 30, 62]. Let $k_{ij}(t)$ be the coupling strength between i-th and j-th oscillators. We study the following Kuramoto system with adaptive coulping:

$$\begin{aligned} \dot{\theta}_i &= \Omega_i + \sum_{j=1}^{N} k_{ij} \sin(\theta_j - \theta_i), \quad t > 0, \quad i, j = 1, \cdots, N, \\ \dot{k}_{ij} &= \mu \Gamma(\theta_j - \theta_i) - \gamma k_{ij}, \end{aligned} \tag{7.0.1}$$

with initial data

$$\theta_i(0) = \theta_i^0, \quad k_{ij}(0) = k_{ij}^0.$$

Here, Γ is a feedback law satisfying symmetry and periodicity:

$$\Gamma(\theta) = \Gamma(-\theta), \quad \Gamma(\theta + 2\pi) = \Gamma(\theta), \quad \theta \in \mathbb{R} \tag{7.0.2}$$

Since the R.H.S. of (7.0.1) is 2π-periodic, (7.0.1) is a dynamical system on N-tori $\mathbb{S}^N$. However, we can lift the dynamical system as a dynamical system on the Euclidean space $\mathbb{R}^N$. Thus, the phase θ_i is assumed to take a value in $\mathbb{R}$ instead of taking mod 2π.

Note that system (7.0.1) cover the classical Kuramoto model for the special case:

$$\Gamma = 0, \quad \gamma = 0, \quad \text{and} \quad k_{ij}^0 = \frac{k}{N}, \quad 1 \le i, j \le N,$$

i.e., system (7.0.1) reduces to the globally coupled Kuramoto model [48, 49]:

$$\dot{\theta}_i = \Omega_i + \frac{k}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \quad t > 0, \quad 1 \le i, j \le N.$$

It is well-known [2, 26] that the Kuramoto model exhibits a spontaneous phase transition from disordered states to ordered states as the coupling strength k increases from zero to infinity, and spontaneous synchronous dynamics emerge in the ensemble of Kuramoto oscillators. However, this plausible scenario has not been completely understood from a rigorous mathematical viewpoint, although there are several recent mathematical studies available [12, 16, 17, 19, 24, 25, 26, 31, 36, 38, 40, 45, 69]. This chapter is base on the joint work [42]

7.1 Adaptive coupling

In this section, we present elementary estimates that will be used in later sections. We also provide a brief summary of our frameworks and main results.

7.1.1 Elementary estimates

In this subsection, we study general properties of a Kuramoto type model (7.0.1) with adaptive couplings. Below, we study the invariance of symmetry and componentwise nonnegativity in the coupling matrix K.

Lemma 7.1.1. *Let* $(\Theta, K) \in \mathbb{R}^N \times \mathbb{R}^{N^2}$ *be a solution to system* (7.0.1)–(7.0.2) *with nonnegative initial coupling strengths*

$$k_{ij}^0 = k_{ji}^0 \geq 0, \quad 1 \leq i, j \leq N.$$

Then, the coupling matrix $K = [k_{ij}]$ *is componentwise nonnegative and symmetric:*

$$k_{ij}(t) = k_{ji}(t) \geq 0, \quad 1 \leq i, j \leq N, \ t > 0.$$

Proof. (i) (Nonnegativity): Suppose that the coupling strength matrix K satisfies the following: for fixed $i, j \in \{1, \cdots, N\}$, suppose that there exists a positive time $t_* \geq 0$ such that

$$k_{ij}(t_*) = 0.$$

It follows from (7.0.1) that

$$\dot{k}_{ij}(t_*) = \Gamma\big(\theta_j(t_*) - \theta_i(t_*)\big) - k_{ij}(t_*) = \Gamma\big(\theta_j(t_*) - \theta_i(t_*)\big) \geq 0.$$

Thus, k_{ij} is nondecreasing at $t = t_*$. Therefore, k_{ij} can not be negative for all t.

(ii) (Preservation of symmetry): Consider the difference $\Delta_{ij}(k) = k_{ij} - k_{ji}$. It is easy to verify that $\Delta_{ij}(k)$ satisfies

$$\frac{d}{dt}\Delta_{ij}(k) = -\gamma \Delta_{ij}(k), \quad t > 0, \qquad \Delta_{ij}(k)(0) = 0,$$

where the even parity of Γ is used. Thus, we have

$$\Delta_{ij}(k) = 0, \quad t > 0, \qquad \text{i.e.,} \qquad k_{ij}(t) = k_{ji}(t).$$

□

We next study the temporal evolution of total phase.

Lemma 7.1.2. *Let* (Θ, K) *be a solution to* (7.0.1) *with natural frequencies and the initial coupling strengths satisfying*

$$\sum_{i=1}^N \Omega_i = 0 \quad \textit{and} \quad k_{ij}^0 = k_{ji}^0, \quad 1 \leq i, j \leq N.$$

Then, the total phase is conserved along the Kuramoto flow (7.0.1)–(7.0.2)*:*

$$\sum_{i=1}^{N} \theta_i(t) = \sum_{i=1}^{N} \theta_i^0, \qquad t \geq 0.$$

Proof. We first note that Lemma 7.1.1 implies

$$k_{ij} = k_{ji}, \quad 1 \leq i, j \leq N.$$

It follows from (7.0.1) and the odd parity of $k_{ij} \sin(\theta_j - \theta_i)$ in $i \leftrightarrow j$ that

$$\frac{d}{dt} \sum_{i=1}^{N} \theta_i = \sum_{i=1}^{N} \Omega_i - \sum_{i,j=1}^{N} k_{ij} \sin(\theta_j - \theta_i) = \sum_{i=1}^{N} \Omega_i = 0.$$

□

Without loss of generality, throughout the rest of this chapter, we assume that the coupling strengths are symmetric, and the average of the natural frequencies is zero, i.e.,

$$k_{ij}(t) = k_{ji}(t), \quad \sum_{i=1}^{N} \Omega_i = 0, \quad \text{and} \quad \sum_{i=1}^{N} \theta_i(t) = 0, \quad t \geq 0.$$

For a given phase configuration Θ, we set extremal indices M and m as follows:

$$\theta_M := \max_{1 \leq i \leq N} \theta_i, \qquad \theta_m := \min_{1 \leq i \leq N} \theta_i, \qquad D(\Theta) := \theta_M - \theta_m.$$

We next show that there exists a trapping set for the flow (7.0.1).

Lemma 7.1.3. (Existence of a trapping set) *Suppose that the natural frequency vector Ω and initial phase configuration Θ^0 satisfy*

$$\Omega = (\Omega_1, \cdots, \Omega_N) = 0, \qquad D(\Theta^0) < \pi.$$

Then for any dynamic solution Θ of (7.0.1)*,*

$$D(\Theta(t)) \leq D(\Theta^0), \quad t \geq 0.$$

Proof. First, set

$$\mathcal{T} := \{t \in [0,\infty) \ : \ D(\Theta(t)) < \pi\}, \quad T^\infty := \sup \mathcal{T}.$$

Then, since $D(\Theta^0) < \pi$ and by the continuity of $D(\Theta(t))$ with respect to t, there exists a $\delta > 0$ such that

$$[0,\delta) \subset \mathcal{T}.$$

We now claim that

$$T^\infty = \infty.$$

Proof of claim: Suppose $T^\infty < \infty$. Then on the finite interval $[0, T^\infty)$, it is easy to see from the dynamics of θ_i that the extremal phases θ_M and θ_m are nonincreasing and nondecreasing, respectively. More precisely,

$$\begin{aligned} \dot\theta_M &= \frac{1}{N}\sum_{j=1}^N k_{Mj}\sin(\theta_j - \theta_M) \le 0, \quad t \in [0, T^\infty), \\ \dot\theta_m &= \frac{1}{N}\sum_{j=1}^N k_{mj}\sin(\theta_j - \theta_m) \ge 0, \quad t \in [0, T^\infty), \end{aligned}$$

where we use the fact that

$$-\pi < \theta_j - \theta_M \le 0, \quad 0 \le \theta_j - \theta_m < \pi.$$

Thus, we have

$$D(\Theta(t)) \le D(\Theta^0) < \pi, \quad t \in [0, T^\infty).$$

Taking the left limit as $t \to T^\infty$ yields

$$D(\Theta(T^\infty)) \le D(\Theta^0) < \pi.$$

Thus, $T^\infty \in \mathcal{T}$, and by the continuity of $D(\Theta(\cdot))$, there exists a $\delta' > 0$ such that

$$T^\infty + \delta' \in \mathcal{T}.$$

This contradicts the fact that T^∞ is the supremum of the set $\mathcal{T}$. Hence,

$$T^\infty = \infty, \quad \text{and} \quad D(\Theta(t)) \le D(\Theta^0), \quad t \ge 0.$$

$\square$

Finally, we present an estimate saying that as long as phases are confined in the half circle, then maximal phases are confined in some interval asymptotically.

Lemma 7.1.4. *For $T \in (0, \infty)$, let (Θ, K) be a solution to the coupled system (7.0.1)–(7.0.2) satisfying a priori assumption on the phase diameter*

$$\sup_{0 \le t < T} D(\Theta(t)) \le D_0 < \pi.$$

Then,we have

$$\|\Theta(t)\| \le \frac{\|\Omega\|}{NR_0k_m} + \Big(\|\Theta^0\| - \frac{\|\Omega\|}{NR_0k_m}\Big)e^{-NR_0k_mt}, \quad 0 \le t < T,$$

where $R_0 := \frac{\sin D_0}{D_0}$

Proof. Multiplying (7.0.1) by $2\theta_i$, summing over i, and using the symmetry $k_{ij} = k_{ji}$ yields

$$\begin{aligned}
\frac{d}{dt}\sum_{i=1}^{N}\theta_i^2 &= 2\sum_i \Omega_i\theta_i + 2\sum_{i,j=1}^{N} k_{ij}\theta_i \sin(\theta_j - \theta_i) \\
&= 2\sum_i \Omega_i\theta_i - 2\sum_{i,j=1}^{N} k_{ij}\theta_j \sin(\theta_j - \theta_i) \qquad (7.1.1)\\
&= 2\sum_i \Omega_i\theta_i - 2\sum_{i,j=1}^{N} k_{ij}(\theta_j - \theta_i)\sin(\theta_j - \theta_i).
\end{aligned}$$

To derive the desired upper bound, we use

$$\begin{aligned}
&(\theta_j - \theta_i)\sin(\theta_j - \theta_i) \ge R_0(\theta_j - \theta_i)^2 \quad \text{and} \\
&|\sin(\theta_i - \theta_j)|^2 \le |\theta_i - \theta_j|^2 \le (|\theta_i| + |\theta_j|)^2 \le 2(|\theta_i|^2 + |\theta_j|^2) \le 2\|\Theta\|^2.
\end{aligned}$$

Thus, it follows from (7.1.1) that

$$\frac{d}{dt}\sum_{i=1}^{N}\theta_i^2 \;\le\; 2\sum_i \Omega_i\theta_i - R_0\sum_{i,j=1}^{N} k_{ij}|\theta_j - \theta_i|^2$$

$$
\begin{aligned}
&\leq \; 2\|\Omega\|\|\Theta\| - R_0 k_m \sum_{i,j=1}^{N} |\theta_j - \theta_i|^2 \\
&= \; 2\|\Omega\|\|\Theta\| - 2NR_0 k_m \sum_{i=1}^{N} \theta_i^2 \\
&= \; 2\|\Omega\|\|\Theta\| - 2NR_0 k_m \|\Theta\|^2.
\end{aligned}
$$

This yields the desired estimate.

□

7.1.2 Frameworks and main results

In this subsection, we briefly summarize our main frameworks and results for an adaptive coupling laws for the dynamics of k_{ij} with Γ_s in [1, 62]:

$$\Gamma_s(\theta) = |\sin\theta|, \quad \theta \in \mathbb{R}.$$

In this case, system (7.0.1) with Γ_s becomes

$$
\begin{cases}
\dot{\theta}_i = \Omega_i + \displaystyle\sum_{j=1}^{N} k_{ij} \sin(\theta_j - \theta_i), \quad t > 0, \quad 1 \leq i, j \leq N, \\
\dot{k}_{ij} = \mu |\sin(\theta_j - \theta_i)| - \gamma k_{ij}.
\end{cases}
\tag{7.1.2}
$$

The coupling term $|\sin(\theta_j - \theta_i)| \approx |\theta_j - \theta_i|$ assumes small values when $|\theta_j - \theta_i|$ so that the growth of coupling strength k_{ij} is suppressed as phase synchronization occurs. Thus, the synchronization estimates are very delicate. Even so, we can use Lyapunov type functional approach to derive asymptotic synchronization estimates for identical and non-identical oscillator systems.

Theorem 7.1.1. (Identical oscillators) *Let $\Theta = \Theta(t)$ be the global smooth solution to* (7.1.2) *satisfying*

$$\Omega = 0 \in \mathbb{R}^N \quad \textit{and} \quad D(\Theta^0) < \frac{\pi}{2}.$$

Then, we have ACS.

$$\lim_{t\to\infty} D(\Theta(t)) = 0, \qquad \lim_{t\to\infty} D(\dot{\Theta}(t)) = 0, \qquad \lim_{t\to\infty} \max_{1\leq i,j\leq N} |k_{ij}(t)| = 0.$$

and

Theorem 7.1.2. (Nonidentical oscillators) *Let* $\Theta = \Theta(t)$ *be a solution to* (7.1.2) *with the a priori assumption*

$$\sup_{0 \le t < \infty} D(\Theta(t)) < \frac{\pi}{2}.$$

Then, we have ACS.

$$\lim_{t \to \infty} |D(\dot{\Theta})(t)| = 0, \quad 1 \le i, j \le N.$$

7.1.3 Discussion on related works

In this subsection, we briefly discuss some relevant literature on the co-evolving dynamics with adpative law Γ_s in a weight network of phase oscillators. We discuss the phase model with adaptive rule Γ_s. In [1], Aoki and Aoyagi proposed a co-evolving phase model with frustrations:

$$\dot{\theta}_i = 1 - \frac{1}{N} \sum_{j=1}^{N} k_{ij} \sin(\theta_i - \theta_j + \alpha), \qquad \dot{k}_{ij} = -\varepsilon \sin(\theta_i - \theta_j + \beta), \qquad |k_{ij}| \le 1,$$

where ε is the positive constant, and α, β are the interaction frustrations. Depending on the nature of the evolution of the coupling weight, the above system can exhibit three types of dynamical behavior: a two-cluster state, a coherent state with a fixed phase relation and a chaotic state with frustration. These observations have been studied analytically for a two-oscillator system and numerically for a many-oscillator system. On the other hand, Ren and Zhao [62] considered the phase model:

$$\dot{\theta}_i = \Omega_i - \frac{1}{N} \sum_{j=1}^{N} k_{ij} \sin(\theta_i - \theta_j), \qquad \dot{k}_{ij} = \varepsilon(|\sin(\beta(\theta_i - \theta_j))| - k_{ij}).$$

Based on numerical simulations, authors shows that the system dynamics approaches to the optimal coupling scheme in the sense of least average coupling cost in all-to-all couplings and nearest neighbor ring topology.

As presented in Subsection 7.1.2, our main results deal with explicit sufficient conditions on the initial phase and coupling strength leading to asymptotic complete synchronization or asymptotic practical synchronization for the Kuramoto phase model with adaptive coupling rule Γ_s and relaxation process such as decay rate toward the phase-locked states. Thus, our results are clearly different from [1, 62].

7.2 Synchronization of a two-body system

In this section, we first consider the adaptive coupling system with $N = 2$, which is analytically treatable unlike many-body systems where $N \geq 3$. and then

We consider a two-oscillator case with $\Omega_2 > \Omega_1$:

$$\begin{aligned} \dot{\theta}_1 &= \Omega_1 + k_{12}\sin(\theta_2 - \theta_1), \quad t > 0, \\ \dot{\theta}_2 &= \Omega_2 + k_{21}\sin(\theta_1 - \theta_2), \\ \dot{k}_{12} &= \mu|\sin(\theta_2 - \theta_1)| - \gamma k_{12}. \end{aligned} \tag{7.2.1}$$

To simplify system (7.2.1), we set

$$k := k_{12}, \quad \theta := \theta_2 - \theta_1, \quad \Omega := \Omega_2 - \Omega_1 > 0.$$

In this case, system (7.2.1) becomes

$$\begin{aligned} \dot{\theta} &= \Omega - 2k\sin\theta, \quad t > 0, \\ \dot{k} &= \mu|\sin\theta| - \gamma k. \end{aligned} \tag{7.2.2}$$

Again, system (7.2.2) can be rewritten as an integro-differential equation:

$$\dot{\theta} = \Omega - 2k^0 e^{-\gamma t}\sin\theta - 2\mu\Big[\int_0^t |\sin\theta(s)|e^{\gamma(s-t)}ds\Big]\sin\theta, \quad t > 0.$$

Note that the differences $\theta = \theta_1 - \theta_2$, $\Omega = \Omega_1 - \Omega_2$, and $k = k_{12}$ satisfy

$$\begin{aligned} \dot{\theta} &= \Omega - 2k\sin\theta, \quad t > 0, \\ \dot{k} &= \mu|\sin\theta| - \gamma k. \end{aligned} \tag{7.2.3}$$

7.2.1 Identical oscillators

In this subsection, we first consider the simple case when two oscillators are identical in the sense that

$$\Omega_1 = \Omega_2, \quad \text{i.e.,} \quad \Omega = 0.$$

Thus, system (7.2.2) becomes

$$\dot{\theta} = -2k \sin\theta, \qquad \dot{k} = \mu |\sin\theta| - \gamma k, \qquad t > 0. \tag{7.2.4}$$

It is easy to verify that the only equilibrium solution to (7.2.4) is $(0, 0)$. Next, we show that this equilibrium is, in fact, asymptotically stable.

Proposition 7.2.1. (Asymptotic stability) *Let (θ, k) be a solution to system (7.2.4) with initial data satisfying*

$$0 < \theta^0 < \pi, \quad k^0 > 0.$$

Then, we have

$$\lim_{t\to\infty} \theta(t) = 0, \qquad \lim_{t\to\infty} k(t) = 0.$$

Proof. First, note that the first equation of (7.2.4) yields

$$0 \le \theta(t) \le \theta^0 < \pi, \quad t \ge 0. \tag{7.2.5}$$

Thus, in this regime, $|\sin\theta| = \sin\theta$ and system (7.2.4) become

$$\dot{\theta} = -2k \sin\theta, \qquad \dot{k} = \mu \sin\theta - \gamma k, \quad t > 0. \tag{7.2.6}$$

Note that the coupling strength always is nonnegative:

$$k(t) \ge 0, \quad t > 0.$$

We first multiply the first equation of (7.2.6) by 2θ and use (7.2.5) and

$$\frac{\sin\theta^0}{\theta^0}\theta \le \sin\theta \le \theta, \quad t \ge 0,$$

to obtain

$$-2k\theta^2 \le \frac{d\theta^2}{dt} \le -2k\frac{\sin\theta^0}{\theta^0}\theta^2.$$

Gronwall's inequality implies

$$\theta^0 \exp\Big[-\int_0^t k(s)ds\Big] \le \theta(t) \le \theta^0 \exp\Big[-\frac{\sin\theta^0}{\theta^0}\int_0^t k(s)ds\Big]. \tag{7.2.7}$$

Note that (7.2.7) yields

$$\lim_{t\to\infty}\theta(t) = 0 \qquad \Longleftrightarrow \qquad \int_0^\infty k(s)ds = \infty. \tag{7.2.8}$$

On the other hand, since θ^2 is a nonincreasing function of t and bounded below by 0,

$$\exists\ \theta^* := \lim_{t\to\infty}\theta(t).$$

We now claim that

$$\theta^* = 0. \tag{7.2.9}$$

Proof of claim (7.2.9): Suppose to the contrary that $\theta^* \neq 0$. Then it follows from (7.2.8) that

$$\theta^* > 0, \quad \text{or equivalently,} \quad \int_0^\infty k(s)ds < \infty. \tag{7.2.10}$$

On the other hand, it follows from the dynamics of k that

$$\dot{k} = \mu\sin\theta - \gamma k \ge \mu\sin\theta^* - \gamma k.$$

Again, we have

$$k(t) \ge \Big(k^0 - \frac{\mu\sin\theta^*}{\gamma}\Big)e^{-\gamma t} + \frac{\mu\sin\theta^*}{\gamma}, \quad t \ge 0. \tag{7.2.11}$$

Integrating (7.2.11) from $t = 0$ to $t = \infty$ yields

$$\int_0^\infty k(s)ds = \infty,$$

which is contradictory to (7.2.10). Hence, we obtain the desired result. $\square$

Remark 7.2.1. *The equilibrium* $(\theta, k) = (0, 0)$ *is not linearly asymptotically stable, which can be directly inferred. Consider a linearized system of* (7.2.4) *near* $(0, 0)$*:*

$$\dot{\theta} = 0, \quad \dot{k} = \mu\theta - \gamma k, \quad t > 0.$$

By direct calculations, we have

$$\theta(t) = \theta^0, \quad k(t) = \Big(k^0 - \frac{\mu\theta^0}{\gamma}\Big)e^{-\gamma t} + \frac{\mu\theta^0}{\gamma}, \quad t > 0.$$

Thus,

$$\lim_{t\to\infty}(\theta(t), k(t)) = \Big(\theta^0, \frac{\mu\theta^0}{\gamma}\Big).$$

Hence, synchronization is a nonlinear effect.

In Proposition 7.2.1 we studied the emergence of synchronization without detailed decay estimates. Next, we study the relaxation estimates of forward synchronization.

Proposition 7.2.2. (Short-time estimate) *Let* (θ, k) *be a solution to system* (7.2.4) *with initial data satisfying*

$$0 < \theta^0 < \pi, \quad k^0 > 0.$$

Then for any $\alpha \in (0, k^0)$*, there exists a positive time* $t_* = t_*(\alpha)$ *and* $C_* = C_*(k^0, \theta^0, \alpha)$ *such that*

$$|\theta(t)| \le |\theta^0|e^{-\alpha R_0 t}, \qquad k(t) \le C_* e^{-\min\{\alpha R_0, \gamma\}t}, \quad t \le t_*(\alpha),$$

where $R_0 := \dfrac{\sin\theta^0}{\theta^0}$

Proof. (i) Note that Proposition 7.2.1 implies that there exists a $t_*(\alpha) > 0$ such that

$$k(t) \ge \alpha, \quad t \in [0, t_*(\alpha)).$$

Together, (7.2.4) and (7.2.5) imply

$$\dot{\theta} = -k\sin\theta \le -\alpha R_0\theta, \quad t \le t_*(\alpha).$$

This yields the desired exponential decay estimate:

$$|\theta(t)| \le |\theta^0| e^{-\alpha R_0 t}, \quad t \le t_*(\alpha).$$

(ii) It follows from the equation for k in (7.2.4) that

$$\dot{k} = \mu|\sin\theta| - \gamma k \le \mu|\theta| - \gamma k \le \mu|\theta^0|e^{-\alpha R_0 t} - \gamma k, \quad t \le t_*(\alpha).$$

Gronwall's lemma yields the desired decay estimate for $k(t)$. $\square$

Below, we study the large-time decay estimates of θ and k, which require the following lemma.

Lemma 7.2.1. *Let* (θ, k) *be a solution to system* (7.2.4) *with initial data satisfying*

$$0 < \theta^0 < \pi, \quad k^0 > 0.$$

Then,

$$\lim_{t\to\infty} \frac{k(t)}{\theta(t)} = \frac{\mu}{\gamma}.$$

Proof. Consider the following system:

$$\dot{\theta} = -2k\sin\theta, \qquad \dot{k} = \mu|\sin\theta| - \gamma k, \qquad t > 0. \tag{7.2.12}$$

It follows from Proposition 7.2.1 that for $\varepsilon \ll 1$, there exists $t^* = t^*(\varepsilon) > 0$ such that

$$(1-\varepsilon)\theta \le \sin\theta \le (1+\varepsilon)\theta, \quad t \ge t^*. \tag{7.2.13}$$

Now, we consider the derivative of $\frac{k}{\theta}$:

$$\begin{aligned} \left(\frac{k}{\theta}\right)' &= \frac{\dot{k}\theta - k\dot{\theta}}{\theta^2} = \frac{(\mu\sin\theta - \gamma k)\theta - k(-2k\sin\theta)}{\theta^2} \\ &= \mu\frac{\theta\sin\theta}{\theta^2} - \gamma\frac{k}{\theta} + \frac{2k^2\sin\theta}{\theta^2}. \end{aligned} \tag{7.2.14}$$

• Case A (Lower bound): For $t \ge t^*$, we use the estimate $\sin\theta \ge (1-\varepsilon)\theta$ to obtain

$$\left(\frac{k}{\theta}\right)' \ge \mu(1-\varepsilon) - \gamma\frac{k}{\theta} + (1-\varepsilon)\frac{2k^2}{\theta}, \quad t \ge t^*.$$

Setting

$$Z := \frac{k}{\theta},$$

Z satisfies

$$Z' \geq \mu(1-\varepsilon) + \Big(2(1-\varepsilon)k - \gamma\Big)Z \geq \mu(1-\varepsilon) - \gamma Z, \quad t \geq t^*.$$

This yields

$$Z(t) \geq \frac{\mu(1-\varepsilon)}{\gamma} + \Big[Z(t^*) - \frac{\mu(1-\varepsilon)}{\gamma}\Big]e^{-\gamma(t-t^*)}, \quad t \geq t^*.$$

Letting $t \to \infty$, we obtain

$$\liminf_{t\to\infty} Z(t) \geq \frac{\mu(1-\varepsilon)}{\gamma}. \tag{7.2.15}$$

- Case B (Upper bound): We use (7.2.13) and (7.2.14) to derive

$$Z' \leq \mu(1+\varepsilon) + \Big(2(1+\varepsilon)k - \gamma\Big)Z, \quad t \geq t^*. \tag{7.2.16}$$

By Proposition 7.2.1, there exists $\tilde{t}^* > t^*$ such that

$$k < \frac{\varepsilon}{2(1+\varepsilon)}\gamma, \qquad t \geq \tilde{t}^*. \tag{7.2.17}$$

Combining (7.2.16) and (7.2.17) yields

$$Z' \leq \mu(1+\varepsilon) - (1-\varepsilon)\gamma Z, \quad t \geq \tilde{t}^*.$$

If follows from the same argument as the lower bound that

$$\limsup_{t\to\infty} Z(t) \leq \Big(\frac{\mu}{\gamma}\Big)\frac{1+\varepsilon}{1-\varepsilon}. \tag{7.2.18}$$

Letting $\varepsilon \to 0$ in (7.2.15) and (7.2.18) yields

$$\frac{\mu}{\gamma} \leq \liminf_{t\to\infty} Z(t) \leq \limsup_{t\to\infty} Z(t) \leq \frac{\mu}{\gamma},$$

which gives the desired result. $\square$

Proposition 7.2.3. *Let (θ, k) be a solution to system (7.2.4) with initial data satisfying*

$$0 < \theta^0 < \pi, \quad k^0 > 0.$$

Then there exists a sufficiently large $\bar{t}^*$ such that θ and k decay to zero like $\frac{1}{t+1}$ for $t > \bar{t}^*$.

Proof. • Case A (Temporal-decay estimate of θ): By Lemma 7.2.1, for any small $\varepsilon \ll 1$, there exists $\hat{t}^* \gg 1$ such that

$$(1-\varepsilon)\frac{\mu}{\gamma} \le \frac{k}{\theta} \le (1+\varepsilon)\frac{\mu}{\gamma}, \quad (1-\varepsilon)\theta \le \sin\theta \le (1+\varepsilon)\theta, \quad t \ge \hat{t}^*. \quad (7.2.19)$$

The first equations of (7.2.12) and (7.2.19) imply

$$-\frac{2(1+\varepsilon)^2\mu}{\gamma}\theta^2 \le \dot{\theta} \le -\frac{2(1-\varepsilon)^2\mu}{\gamma}\theta^2, \quad t \ge \hat{t}^*.$$

This yields

$$\frac{\gamma\theta(\hat{t}^*)}{\gamma + 2(1+\varepsilon)^2\mu\theta(\hat{t}^*)(t-\hat{t}^*)} \le \theta(t) \le \frac{\gamma\theta(\hat{t}^*)}{\gamma + 2(1-\varepsilon)^2\mu\theta(\hat{t}^*)(t-\hat{t}^*)}, \qquad t \ge \hat{t}^*.$$

• Case B (Temporal-decay estimate of k): We use (7.2.19) to see that

$$k(t) \le (1+\varepsilon)\frac{\mu}{\gamma}\theta \le (1+\varepsilon)\frac{\mu}{\gamma}\frac{\gamma\theta(\hat{t}^*)}{\gamma + 2(1-\varepsilon)^2\mu\theta(\hat{t}^*)(t-\hat{t}^*)}, \quad t \ge \hat{t}^*. \quad (7.2.20)$$

On the other hand,

$$k(t) \ge (1-\varepsilon)\frac{\mu}{\gamma}\theta \ge (1-\varepsilon)\frac{\mu}{\gamma}\frac{\gamma\theta(\hat{t}^*)}{\gamma + 2(1+\varepsilon)^2\mu\theta(\hat{t}^*)(t-\hat{t}^*)}, \quad t \ge \hat{t}^*. \quad (7.2.21)$$

Finally, by combining (7.2.20) and (7.2.21), we derive the desired result. □

Remark 7.2.2. *It follows from Propositions 7.2.2 and 7.2.3 that θ and k exhibit two stages in the relaxation process: a fast exponential relaxation stage for small-time and a slow algebraic relaxation stage for large-time. For a constant coupling strength $k(t) = k^\infty$ and generic initial data, it is well-known that relaxation toward complete synchronization is always exponential [31]. Thus, competition between θ and k makes the relaxation asymptotically slow.*

7.2.2 Nonidentical oscillators

In this subsection, we study complete synchronization of a two-body system with non-identical natural frequencies $\Omega_1 \neq \Omega_2$.

We first look for equilibria $(\theta^\infty, k^\infty)$ of (7.2.3):

$$\Omega - 2k^\infty \sin\theta^\infty = 0, \quad \mu|\sin\theta^\infty| - \gamma k^\infty = 0, \quad \Omega > 0.$$

This yields

$$k^\infty = \frac{\mu}{\gamma}|\sin\theta^\infty|, \qquad |\sin\theta^\infty|\sin\theta^\infty = \frac{\gamma\Omega}{2\mu}. \tag{7.2.22}$$

Note that system (7.2.22) is solvable if and only if

$$\theta^\infty \in [0,\pi], \qquad \Omega \le \frac{2\mu}{\gamma}.$$

In this case, equilibria are determined as follows.

If $\Omega < \frac{2\mu}{\gamma}$, then there are two equilbria:

$$(\theta^{1\infty}, k^{1\infty}) = \Big(\arcsin\sqrt{\frac{\gamma\Omega}{2\mu}}, \sqrt{\frac{\mu\Omega}{2\gamma}}\Big), \quad (\theta^{2\infty}, k^{2\infty}) = \Big(\pi - \arcsin\sqrt{\frac{\gamma\Omega}{2\mu}}, \sqrt{\frac{\mu\Omega}{2\gamma}}\Big).$$

If $\Omega = \frac{2\mu}{\gamma}$,

$$(\theta^{3\infty}, k^{3\infty}) = \Big(\frac{\pi}{2}, \frac{\mu}{\gamma}\Big).$$

We now perform a linear stability analysis for system (7.2.2). First, we set

$$\bar\theta := \theta - \theta^\infty, \quad \bar k := k - k^\infty.$$

Then the linearized system for $(\bar\theta, \bar k)$ is given by

$$\frac{d}{dt}\begin{pmatrix} \bar\theta \\ \bar k \end{pmatrix} = \begin{pmatrix} -k^\infty\cos\theta^\infty & -\sin\theta^\infty \\ \mu\cos\theta^\infty & -\gamma \end{pmatrix}\begin{pmatrix} \bar\theta \\ \bar k \end{pmatrix}. \tag{7.2.23}$$

By direct calculations, the characteristic polynomial of the coefficient matrix of (7.2.23) is

$$p_1(\lambda) = \lambda^2 + (\gamma + k^\infty\cos\theta^\infty)\lambda + \gamma k^\infty\cos\theta^\infty + \mu\sin\theta^\infty\cos\theta^\infty. \tag{7.2.24}$$

Now, let λ_1 and λ_2 be the roots of (7.2.24), i.e., the eigenvalues of the coefficient matrix in (7.2.23). To determine the sign of the real parts of λ_i, we consider two cases.

- Case A (supercritical case, $\Omega < \frac{2\mu}{\gamma}$): For $(\theta^{1\infty}, k^{1\infty})$, since $\cos\theta^{1\infty} > 0$, it follows that

$$\begin{aligned}
&\lambda_1 + \lambda_2 = -(\gamma + k^{\infty}\cos\theta^{\infty}) < 0,\\
&\lambda_1\lambda_2 = \gamma k^{\infty}\cos\theta^{\infty} + \mu\sin\theta^{\infty}\cos\theta^{\infty} > 0.
\end{aligned} \tag{7.2.25}$$

Since the polynomial $p_1(\lambda)$ has real coefficients, it has two real roots or two conjugate complex roots. Thus, (7.2.25) implies that the real parts of the two eigenvalues are negative in both cases. Hence, $(\theta^{1\infty}, k^{1\infty})$ is a linearly stable hyperbolic equilibrium.

On the other hand, for $(\theta^{2\infty}, k^{2\infty})$, since $\cos\theta^{2\infty} < 0$, it follows that

$$\lambda_1\lambda_2 = \gamma k^{\infty}\cos\theta^{\infty} + \mu\sin\theta^{\infty}\cos\theta^{\infty} = \cos\theta^{\infty}(\gamma k^{\infty} + \mu\sin\theta^{\infty}) < 0.$$

This implies that the polynomial $p_1(\lambda)$ does not have two conjugate complex roots, and the linearized system (7.2.23) has one positive eigenvalue and one negative eigenvalue. Hence, $(\bar\theta_2, \bar k)$ is linearly unstable.

- Case B (critical case, $\Omega = \frac{2\mu}{\gamma}$) : In this case, only one equilibrium exists:

$$(\theta^{3\infty}, k^{3\infty}) = \Big(\frac{\pi}{2}, \frac{\mu}{\gamma}\Big).$$

The corresponding linearized system at $(\theta^{3\infty}, k^{3\infty})$ is as follows.

$$\dot{\bar\theta} = -\bar k, \qquad \dot{\bar k} = -\gamma\bar k, \quad t > 0.$$

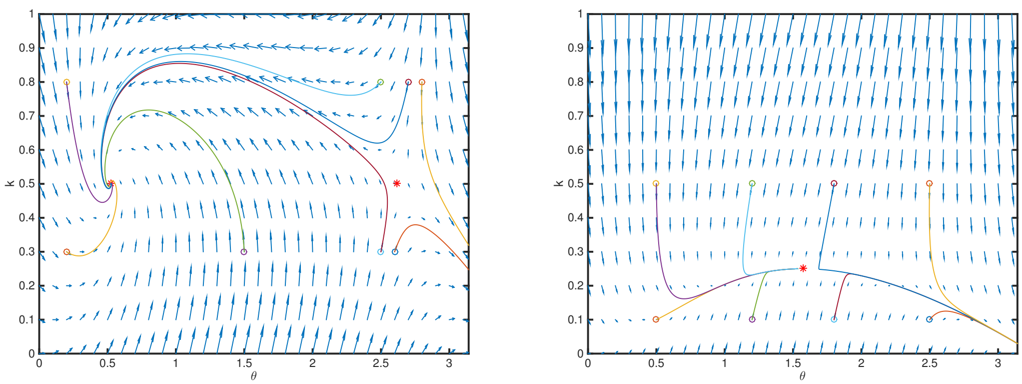
Thus, by direct calculation, we have

$$\bar k(t) = \bar k^0 e^{-\gamma t}, \qquad \bar\theta(t) = \bar\theta^0 - \frac{\bar k^0}{\gamma} + \frac{\bar k^0}{\gamma}e^{-\gamma t}, \quad t \ge 0.$$

Note that

$$\lim_{t\to\infty}(\bar k(t), \bar\theta(t)) = \Big(0, \bar\theta^0 - \frac{\bar k^0}{\gamma}\Big).$$

Thus the unique equilibrium $(\theta^{3\infty}, k^{3\infty})$ is not asymptotically stable. This can be seen easily from two numerical simulations to assess the system's behavior. The results are shown in Figure 7.1.



(a) Phase portrait for the supercritical case

(b) Phase portrait for the critical case

Figure 7.1: (a) $(\mu, \gamma, \Omega) = (1, 1, 0.5)$, $(b)(\mu, \gamma, \Omega) = (1, 4, 0.5)$.

7.3 Synchronization estimate of a many-body system

In this section, we study the complete synchronization problem for the adaptive coupling system with $N \geq 3$. As in the previous section, we provide a rigorous result for complete synchronization of identical oscillators; in contrast, for nonidentical oscillators, we provide complete synchronization under he a priori assumption:

$$\sup_{0 \leq t < \infty} D(\Theta(t)) < \frac{\pi}{2}.$$

7.3.1 Identical oscillators

In this subsection, we present a synchronization estimate for (7.1.2). We assume that

$$\Omega_i = 0, \quad 1 \leq i \leq N.$$

In this case, system (7.1.2) becomes

$$\begin{aligned}\dot{\theta}_i &= \sum_{j=1}^{N} k_{ij}\sin(\theta_j-\theta_i), \quad t>0, \quad 1\le i\le N,\\ \dot{k}_{ij} &= \mu|\sin(\theta_j-\theta_i)| - \gamma k_{ij}.\end{aligned} \tag{7.3.1}$$

We set extremal indices M and m as follows:

$$\theta_M := \max_{1\le i\le N}\theta_i, \qquad \theta_m := \min_{1\le i\le N}\theta_i.$$

For a solution (Θ, K) to (7.3.1), we set three Lyapunov functionals:

$$\begin{aligned}\mathcal{L}_1(t) &:= D(\Theta(t)) + \frac{1}{2\mu}\sum_{j=1}^{N}(k_{Mj}^2 + k_{mj}^2), \quad t\ge 0,\\ \mathcal{L}_2(t) &:= \sum_{i=1}^{N}|\theta_i|^2 + \frac{2\gamma}{3\mu^2}\sum_{i,j=1}^{N}k_{ij}^3,\\ \mathcal{L}_3(t) &:= \sum_{i=1}^{N}|\dot{\theta}_i|^2 + \frac{\mu}{4}\sum_{i,j=1}^{N}\mathrm{sgn}(\theta_i-\theta_j)\Big(2(\theta_i-\theta_j) - \sin 2(\theta_i-\theta_j)\Big).\end{aligned} \tag{7.3.2}$$

In the following lemma, we study the time-variation of the functionals in (7.3.2).

Lemma 7.3.1. *Let (Θ, K) be a solution to (7.3.1) with initial data satisfying $D(\Theta^0) < \frac{\pi}{2}$. Then the functionals in (7.3.2) are nonincreasing along the flow (7.3.1):*

(i) $\mathcal{L}_1(t) + \dfrac{\gamma}{\mu}\displaystyle\sum_{j=1}^{N}\int_0^t \Big(k_{Mj}^2(s) + k_{mj}^2(s)\Big)ds = \mathcal{L}_1(0), \quad t>0.$

(ii) $\mathcal{L}_2(t) + \dfrac{1}{\mu^2}\displaystyle\sum_{i,j=1}^{N}\int_0^t k_{ij}(s)\Big(|\dot{k}_{ij}(s)|^2 + \gamma k_{ij}^2(s)\Big)ds \le \mathcal{L}_2(0).$

(iii) $\mathcal{L}_3(t) + 2\displaystyle\int_0^t \|\dot{\Theta}(s)\|_2^2 ds + \sum_{i,j=1}^{N}\int_0^t k_{ij}(s)\cos(\theta_i-\theta_j)|\dot{\theta}_i - \dot{\theta}_j|^2 ds$
$= \mathcal{L}_3(0).$

Proof. Suppose $D(\Theta^0) < \frac{\pi}{2}$. It follows from Lemma 7.1.3 that

$$D(\Theta(t)) \le D(\Theta^0) < \frac{\pi}{2}, \quad t \ge 0.$$

(i) It follows from (7.3.1) that

$$\frac{d}{dt}D(\Theta(t)) = \dot\theta_M - \dot\theta_m = \sum_{j=1}^N k_{Mj}\sin(\theta_j - \theta_M) - \sum_{j=1}^N k_{mj}\sin(\theta_j - \theta_m). \tag{7.3.3}$$

Since the phase difference $D(\Theta(t)) < \frac{\pi}{2}$, we obtain

$$\begin{aligned}\frac{d}{dt}\sum_{j=1}^N \frac{k_{Mj}^2}{2} &= \sum_{j=1}^N k_{Mj}\dot k_{Mj} = \sum_{j=1}^N k_{Mj}\Big(\mu|\sin(\theta_M - \theta_j)| - \gamma k_{Mj}\Big)\\ &= -\mu\sum_{j=1}^N k_{Mj}\sin(\theta_j - \theta_M) - \gamma\sum_{j=1}^N k_{Mj}^2.\end{aligned} \tag{7.3.4}$$

Similarly, we have

$$\begin{aligned}\frac{d}{dt}\sum_{j=1}^N \frac{k_{mj}^2}{2} &= \sum_{j=1}^N k_{mj}\dot k_{mj} = \sum_{j=1}^N k_{mj}(\mu|\sin(\theta_m - \theta_j)| - \gamma k_{mj})\\ &= \mu\sum_{j=1}^N k_{mj}\sin(\theta_j - \theta_m) - \gamma\sum_{j=1}^N k_{mj}^2.\end{aligned} \tag{7.3.5}$$

The linear combination (7.3.3) + $\frac{1}{\mu} \times \Big((7.3.4) + (7.3.5)\Big)$ yields

$$\frac{d}{dt}\Big[D(\Theta(t)) + \frac{1}{2\mu}\sum_{j=1}^N (k_{Mj}^2 + k_{mj}^2)\Big] = -\frac{\gamma}{\mu}\sum_{j=1}^N (k_{Mj}^2 + k_{mj}^2).$$

This gives the desired result.

(ii) Multiplying (7.3.1) by $2\theta_i$, summing the result, and using the $i \leftrightarrow j$ exchange technique, we obtain

$$\frac{d}{dt}\sum_{i=1}^N \theta_i^2 = \sum_{i,j=1}^N k_{ij}(\theta_i - \theta_j)\sin(\theta_j - \theta_i).$$

Using $x \sin x \geq \sin^2 x$ for $x \in [-\pi, \pi]$, gives rise to the estimate

$$\begin{aligned}\frac{d}{dt}\sum_{i=1}^N \theta_i^2 &= -\sum_{i,j=1}^N k_{ij}(\theta_j - \theta_i)\sin(\theta_j - \theta_i) \\ &\leq -\sum_{i,j=1}^N k_{ij}\sin^2(\theta_j - \theta_i) = -\sum_{i,j=1}^N \frac{k_{ij}}{\mu^2}(\dot{k}_{ij} + \gamma k_{ij})^2 \\ &= -\frac{1}{\mu^2}\sum_{i,j=1}^N k_{ij}|\dot{k}_{ij}|^2 - \frac{2\gamma}{\mu^2}\sum_{i,j=1}^N k_{ij}^2\dot{k}_{ij} - \frac{\gamma^2}{\mu^2}\sum_{i,j=1}^N k_{ij}^3 \\ &= -\frac{1}{\mu^2}\sum_{i,j=1}^N k_{ij}(|\dot{k}_{ij}|^2 + \gamma^2 k_{ij}^2) - \frac{d}{dt}\sum_{i,j=1}^N \frac{2\gamma}{3\mu^2}k_{ij}^3.\end{aligned} \tag{7.3.6}$$

Integrating (7.3.6) yields the desired result.

(iii) We differentiate (7.3.1) to consider the following system for frequency $\dot{\theta}_i$:

$$\begin{aligned}\frac{d}{dt}\dot{\theta}_i &= \sum_{j=1}^N \dot{k}_{ij}\sin(\theta_j - \theta_i) + \sum_{j=1}^N k_{ij}(\dot{\theta}_j - \dot{\theta}_i)\cos(\theta_j - \theta_i) \\ &= \mu\sum_{j=1}^N |\sin(\theta_i - \theta_j)|\sin(\theta_i - \theta_j) - \gamma\dot{\theta}_i - \sum_{j=1}^N k_{ij}(\dot{\theta}_i - \dot{\theta}_j)\cos(\theta_i - \theta_j).\end{aligned} \tag{7.3.7}$$

Multiplying (7.3.7) by $2\dot{\theta}_i$, summing the result, and using the $i \leftrightarrow j$ exchange technique yields

$$\begin{aligned}\frac{d}{dt}\sum_{i=1}^N \dot{\theta}_i^2 &= \mu\sum_{i,j=1}^N (\dot{\theta}_i - \dot{\theta}_j)|\sin(\theta_i - \theta_j)|\sin(\theta_i - \theta_j) \\ &\quad - 2\gamma\sum_{i=1}^N \dot{\theta}_i^2 - \sum_{i,j=1}^N k_{ij}|\dot{\theta}_i - \dot{\theta}_j|^2\cos(\theta_i - \theta_j) \\ &= -\frac{\mu}{4}\frac{d}{dt}\sum_{i,j=1}^N \Big\{\mathrm{sgn}(\theta_i - \theta_j)\Big(2(\theta_i - \theta_j) - \sin 2(\theta_i - \theta_j)\Big)\Big\}\end{aligned}$$

$$- 2\gamma \sum_{i=1}^{N} \dot{\theta}_i^2 - \sum_{i,j=1}^{N} k_{ij} |\dot{\theta}_i - \dot{\theta}_j|^2 \cos(\theta_i - \theta_j), \tag{7.3.8}$$

where the following relation is used:

$$\begin{aligned}
&\frac{d}{dt}\Big\{\mathrm{sgn}(\theta_i - \theta_j)\Big(2(\theta_i - \theta_j) - \sin 2(\theta_i - \theta_j)\Big)\Big\} \\
&\qquad = \mathrm{sgn}(\theta_i - \theta_j)\Big(2(\dot{\theta}_i - \dot{\theta}_j) - 2(\dot{\theta}_i - \dot{\theta}_j)(\cos 2(\theta_i - \theta_j))\Big) \\
&\qquad = 4\mathrm{sgn}(\theta_i - \theta_j)\Big((\dot{\theta}_i - \dot{\theta}_j) \sin^2(\theta_i - \theta_j)\Big) \\
&\qquad = 4(\dot{\theta}_i - \dot{\theta}_j) |\sin(\theta_i - \theta_j)| \sin(\theta_i - \theta_j).
\end{aligned} \tag{7.3.9}$$

Integrating (7.3.8) yields the desired result. □

Proof of Theorem 7.1.1: Let $\Theta = \Theta(t)$ be a solution to (7.3.1) with initial data Θ^0 satisfying $D(\Theta^0) < \frac{\pi}{2}$. Then it follows from Lemma 7.1.3 that

$$\sup_{t \ge 0} D(\Theta(t)) \le D(\Theta^0) < \pi.$$

• Case A: We first provide the estimates:

$$\lim_{t \to \infty} |k_{ij}(t)| = 0, \quad \lim_{t \to \infty} |\dot{\theta}_i(t)| = 0, \quad 1 \le i, j \le N.$$

◇ Case A.1: For the zero convergence of the coupling strength, we use Lemma 7.3.1 to obtain

$$\int_0^\infty k_{ij}^3(s) ds \le \mathcal{L}_2(0) < \infty. \tag{7.3.10}$$

On the other hand, note that

$$\dot{k}_{ij} = \mu |\sin(\theta_j - \theta_i)| - \gamma k_{ij} \le \mu |\sin(\theta_j - \theta_i)| - \gamma k_{ij}.$$

This yields

$$k_{ij}(t) \le k_{ij}(0) e^{-\gamma t} + \frac{\mu}{\gamma}(1 - e^{-\gamma t}) \le k_{ij}(0) + \frac{\mu}{\gamma}.$$

It follows from (7.3.1) that

$$|\dot{k}_{ij}| = \big|\mu |\sin(\theta_j - \theta_i)| - \gamma k_{ij}\big| \le 2\mu + \gamma k_{ij}(0). \tag{7.3.11}$$

Then the estimate (7.3.11) and uniform boundedness of k_{ij} (see (ii) of Lemma 7.3.1) imply the uniform boundedness of $\frac{d}{dt}k_{ij}^3$:

$$\sup_{t\geq 0}\Big|\frac{d}{dt}k_{ij}^3\Big| < \infty. \tag{7.3.12}$$

Finally, we combine (7.3.10) and (7.3.12) to conclude

$$\lim_{t\to\infty} k_{ij}(t) = 0, \quad \text{or equivalently,} \ \lim_{t\to\infty}|k_{ij}(t)| = 0. \tag{7.3.13}$$

This clearly yields

$$\sup_{t\geq 0}\max_{1\leq i,j\leq N} k_{ij}(t) \leq k^\infty.$$

◇ Case A.2: It follows from Lemma 7.3.1 that

$$\begin{aligned}
&\|\dot\Theta(t)\|_2^2 \leq \mathcal{L}_3(t) \leq \mathcal{L}_3(0) \qquad \text{and}\\
&2\int_0^t \|\dot\Theta(s)\|_2^2 ds + \sum_{i,j=1}^N \int_0^t k_{ij}(s)|\dot\theta_i - \dot\theta_j|^2\cos(\theta_i-\theta_j)ds \leq \mathcal{L}_3(0).
\end{aligned} \tag{7.3.14}$$

Recall from (7.3.8) that

$$\begin{aligned}
\frac{d}{dt}\|\dot\Theta\|_2^2 &= \mu\sum_{i,j=1}^N(\dot\theta_i-\dot\theta_j)|\sin(\theta_i-\theta_j)|\sin(\theta_i-\theta_j)\\
&\quad - 2\gamma\sum_{i=1}^N\dot\theta_i^2 - \sum_{i,j=1}^N k_{ij}|\dot\theta_i-\dot\theta_j|^2\cos(\theta_i-\theta_j)\\
&\leq \mu\Big|\sum_{i,j=1}^N|\sin(\theta_i-\theta_j)|\sin(\theta_i-\theta_j)(\dot\theta_i-\dot\theta_j)\Big|\\
&\leq \mu\sum_{i,j=1}^N(1+|\dot\theta_i-\dot\theta_j|^2) \leq \mu\Big(N^2+2\|\dot\Theta\|_2^2\Big)\\
&\leq \mu\Big(N^2+2\mathcal{L}_3(0)\Big).
\end{aligned} \tag{7.3.15}$$

Finally, we combine (7.3.14) and (7.3.15) to obtain

$$\lim_{t\to\infty}\|\dot\Theta(t)\|_2 = 0.$$

• Case B: We first provide the estimate:

$$\lim_{t\to\infty} D(\Theta(t)) = 0.$$

Note that for all $1 \le i, j \le N$,

$$\theta_i(t) - \theta_j(t) \text{ is uniformly continuous for } t \ge 0.$$

Since $D(\dot\Theta) = \dot\theta_M - \dot\theta_m \le 0$, we know that $D(\Theta)$ is nonincreasing and bounded below. Thus, there exists $D^* \ge 0$ such that

$$D^* := \lim_{t\to\infty} D(\Theta(t)).$$

Suppose that D^* is strictly positive, i.e.,

$$\lim_{t\to\infty} (\theta_M - \theta_m) \ge D^* > 0.$$

For each $t > 0$, there is an (i, j) pair such that $|\theta_i(t) - \theta_j(t)| \ge D_*$. On the other hand, because of the uniform continuity of $\theta_i - \theta_j$, there is a positive constant $\delta > 0$ such that $|\theta_i(s) - \theta_j(s)| > \frac{D^*}{2}$ for $s \in (t - \delta, t)$. We estimate that

$$\begin{aligned}
k_{ij}(t) &= k_{ij}(0)e^{-\gamma t} + \mu \int_0^t \big|\sin\big(\theta_i(s) - \theta_j(s)\big)\big| e^{-\gamma(t-s)} ds \\
&\ge \mu \int_{t-\delta}^t \big|\sin\big(\theta_i(s) - \theta_j(s)\big)\big| e^{-\gamma(t-s)} ds \\
&\ge \mu\delta \sin\Big(\frac{D^*}{2}\Big) e^{-\gamma\delta},
\end{aligned}$$

which contradicts (7.3.13). Therefore, we conclude that $D^* = 0$. This completes the proof.

7.3.2 Nonidentical oscillators

We now return to the nonidentical case when $D(\Omega) > 0$. We introduce a fourth Lyapunov functional $\mathcal{L}_4$ as follows. For a global solution (Θ, K) to (7.3.1), we define

$$\mathcal{L}_4(t) := \sum_{i=1}^N \Big(|\dot\theta_i|^2 - 2\gamma\Omega_i\theta_i\Big) + \frac{\mu}{4}\sum_{i,j=1}^N \operatorname{sgn}(\theta_i - \theta_j)\Big(2(\theta_i - \theta_j) - \sin 2(\theta_i - \theta_j)\Big).$$

Lemma 7.3.2. *For Let* $\Theta = \Theta(t)$ *be a global solution to* (7.3.1) *with the a priori assumption:*

$$\sup_{t\geq 0} D(\Theta(t)) < \frac{\pi}{2}.$$

Then, the functional $\mathcal{L}_4(t)$ *satisfies*

$$\mathcal{L}_4(t) + 2\int_0^t \|\dot{\Theta}\|^2 ds + \sum_{i,j=1}^N \int_0^t k_{ij}\cos(\theta_i - \theta_j)|\theta_i - \theta_j|^2 ds = \mathcal{L}_4(0).$$

Proof. We differentiate the first equation of (7.1.2) to consider the following:

$$\begin{aligned}\frac{d}{dt}\dot{\theta}_i &= \sum_{j=1}^N \dot{k}_{ij}\sin(\theta_j - \theta_i) + \sum_{j=1}^N k_{ij}\cos(\theta_j - \theta_i)(\dot{\theta}_j - \dot{\theta}_i)\\ &= \mu\sum_{j=1}^N |\sin(\theta_i - \theta_j)|\sin(\theta_i - \theta_j) - \gamma(\dot{\theta}_i - \Omega_i) \qquad (7.3.16)\\ &+ \sum_{j=1}^N k_{ij}\cos(\theta_i - \theta_j)(\dot{\theta}_i - \dot{\theta}_j).\end{aligned}$$

Multiplying (7.3.16) by $2\dot{\theta}_i$, summing the result, and using the $i \leftrightarrow j$ exchange technique, we obtain

$$\begin{aligned}\frac{d}{dt}\sum_{i=1}^N \dot{\theta}_i^2 &= \mu\sum_{i,j=1}^N (\dot{\theta}_i - \dot{\theta}_j)|\sin(\theta_i - \theta_j)|\sin(\theta_i - \theta_j)\\ &\quad - 2\gamma\Big(\sum_{i=1}^N \dot{\theta}_i^2 - \sum_{i=1}^N \Omega_i\dot{\theta}_i\Big) - \sum_{j=1}^N k_{ij}|\dot{\theta}_i - \dot{\theta}_j|^2\cos(\theta_i - \theta_j)\\ &= -\frac{\mu}{4}\frac{d}{dt}\sum_{i,j=1}^N \Big\{\mathrm{sgn}(\theta_i - \theta_j)\Big(2(\theta_i - \theta_j) - \sin 2(\theta_i - \theta_j)\Big)\Big\}\\ &\quad - 2\gamma\sum_{i=1}^N \dot{\theta}_i^2 - 2\gamma\frac{d}{dt}\sum_{i=1}^N \Omega_i\theta_i - \sum_{j=1}^N k_{ij}|\dot{\theta}_i - \dot{\theta}_j|^2\cos(\theta_i - \theta_j),\end{aligned} \qquad (7.3.17)$$

where (7.3.9) is utilized. Integrating (7.3.17) yields the desired result. □

Proof of Theorem 7.1.2: Let $\Theta = \Theta(t)$ be a solution to (7.3.1) with the a priori assumption $D(\Theta(t)) < \frac{\pi}{2}$. We claim:

$$\lim_{t\to\infty} |\dot{\theta}_i(t)| = 0, \quad 1 \le i, j \le N.$$

With the assumption that $D(\Theta(t)) < \frac{\pi}{2}$ for $t \ge 0$, functional $\mathcal{L}_4(t)$ is positive for all t. By Lemma 7.3.2, we have

$$\begin{aligned} 2\int_0^t \|\dot{\Theta}\|^2 ds + \int_0^t \sum_{i,j=1}^N |\theta_i - \theta_j|^2 \cos(\theta_i - \theta_j) ds \\ = \mathcal{L}_4(0) - \mathcal{L}_4(t) \le |\mathcal{L}_4(0)| + |\mathcal{L}_4(t)|. \end{aligned} \tag{7.3.18}$$

We now show the uniform boundedness of $\mathcal{L}_4(t)$ and $\|\dot{\Theta}\|$. First, we recall the uniform boundedness of coupling strengths k_{ij}:

$$k_{ij}(t) \le \frac{\mu}{\gamma} + \Big(k_{ij}(0) - \frac{\mu}{\gamma}\Big)e^{-\gamma t} \le k^\infty.$$

Since $D(\Theta(t)) < \dfrac{\pi}{2}$ and $\displaystyle\sum_{i=1}^N \theta_i = 0$, we have

$$\sum_{i=1}^N |\theta_i| \le \frac{N\pi}{2}.$$

Thus, we attain

$$\begin{aligned} |\mathcal{L}_4(t)| &\le \sum_{i=1}^N |\dot{\theta}_i|^2 + 2\gamma\|\Omega\|_\infty \sum_{i=1}^N (|\frac{\pi}{2}| + |\theta_i|) + \frac{\mu}{4}\sum_{i,j=1}^N (2D(\Theta(t)) + 1) \\ &\le N(\|\Omega\|_\infty + K^\infty N)^2 + 2\gamma\|\Omega\|_\infty N\pi + \frac{\mu N^2(\pi+1)}{4}. \end{aligned}$$

Hence,

$$\sup_{t\ge 0} |\mathcal{L}_4(t)| \le N(\|\Omega\|_\infty + K^\infty N)^2 + 2\gamma\|\Omega\|_\infty N\pi + \frac{\mu N^2(\pi+1)}{4} := C_1^\infty.$$

This implies

$$\|\dot{\Theta}\|^2 \le \mathcal{L}^4(t) + \sum_{i=1}^N 2\gamma\Omega_i\theta_i \le C_1^\infty + 2\gamma\|\Omega\|\|\theta\| := C_2^\infty.$$

It suffices to show that $\frac{d}{dt}\|\dot\Theta(t)\|^2$ is uniformly bounded:

$$
\begin{aligned}
\frac{d}{dt}\|\dot\Theta\|^2 &= \mu\sum_{i,j=1}^N(\dot\theta_i-\dot\theta_j)|\sin(\theta_i-\theta_j)|\sin(\theta_i-\theta_j)\\
&\quad -2\gamma\sum_{i=1}^N\dot\theta_i^2-2\gamma\sum_{i=1}^N\Omega_i\dot\theta_i-\sum_{i,j=1}^N k_{ij}|\dot\theta_i-\dot\theta_j|^2\cos(\theta_i-\theta_j)\\
&\le \mu\Big|\sum_{i,j=1}^N|\sin(\theta_i-\theta_j)|\sin(\theta_i-\theta_j)(\dot\theta_i-\dot\theta_j)\Big|+2\gamma\sum_{i=1}^N|\Omega_i||\dot\theta_i|\\
&\le \mu\Big(N^2+2\|\dot\Theta\|_2^2\Big)+2\gamma\|\Omega\|\|\dot\Theta\|\\
&\le \mu\Big(N^2+2C_2^\infty\Big)+2\gamma\|\Omega\|\sqrt{C_2^\infty}.
\end{aligned}
\tag{7.3.19}
$$

Therefore, from (7.3.18) and (7.3.19), we conclude that

$$\lim_{t\to\infty}\|\dot\Theta\|_2=0.$$

This completes the proof of Theorem 7.1.2.

Chapter 8

Existence of BV-solution to the Kuramoto-Sakaguchi equation

In this chapter, we consider the kinetic Kuramoto model (2.3.4) which is derived by mean field limit. We present the global existence of BV-solution to the kinetic Kuramoto model using the front tracking method from hyperbolic conservation law. We provide approximate solutions which is piecewise constant with finite number of jump discontinuities. We also show the nonlinear instability of incoherent solution $f(\theta, \Omega, t) = \frac{\delta(\Omega)}{2\pi}$ for identical oscillators. This chapter is based on the joint work in [6]

8.1 Assumptions and basic properties

This is assumed to be a random variable extracted from some given density function $g = g(\Omega)$ satisfying some normalization conditions:

$$\int_{\mathbb{R}} g(\Omega) d\Omega = 1, \quad \int_{\mathbb{R}} \Omega g(\Omega) d\Omega = 0, \quad \operatorname{supp} g(\cdot) \subset\subset \mathbb{R}, \quad g(\Omega) \geq 0, \quad g \in L^1(\mathbb{R}). \tag{8.1.1}$$

Let $\rho = \rho(\theta, t)$ be the density function in θ-variable which corresponds to the θ marginal function of f:

$$\rho(\theta, t) := \int_{\mathbb{R}} f(\theta, \Omega, t) d\Omega, \quad t \geq 0. \tag{8.1.2}$$

Notice that the equation (2.3.4) can be written into a quasilinear form:

$$\partial_t f + \omega[f]\partial_\theta f + f\partial_\theta(\omega[f]) = 0\,,$$

that is

$$\partial_t f + (\Omega - KL[\rho])\partial_\theta f = Kf\partial_\theta L[\rho]$$

or equivalently, it can be rewritten as a characteristic system:

$$\dot{\theta} = \Omega - KL[\rho], \qquad \dot{f} = Kf\partial_\theta L[\rho] \tag{8.1.3}$$

where the convolution term $L[\rho]$ is given by

$$L[\rho](\theta, t) := \int_{\mathbb{T}} \sin(\theta - \theta_*)\rho(\theta_*, t)d\theta_*.$$

We recall conservation laws for the kinetic Kuramoto equation in the following lemma.

Lemma 8.1.1. *Let f be a $\mathcal{C}^1$-solution to* (2.3.4) *with initial datum f_0 satisfying the following condition:*

$$f_0(\theta, \Omega) = f_0(\theta + 2\pi, \Omega), \qquad \int_{\mathbb{T}} f_0(\theta, \Omega)d\theta = g(\Omega), \iint_{\mathbb{T}\times\mathbb{R}} f_0 d\Omega d\theta = 1.$$

Then for $t \geq 0$, we have

$$\int_{\mathbb{T}} f(\theta, \Omega, t)d\theta = g(\Omega), \quad \iint_{\mathbb{T}\times\mathbb{R}} f(\theta, \Omega, t)d\Omega d\theta = 1.$$

Proof. We use the equation (2.3.4) and relations

$$\begin{aligned}
&\omega[f](2\pi, \Omega, t) = \Omega - KL[\rho](2\pi, t) = \Omega - KL[\rho](0, t) = \omega[f](0, \Omega, t),\\
&f(2\pi, \Omega, t) = f(0, \Omega, t), \quad (\Omega, t) \in \mathbb{R} \times \mathbb{R}_+,
\end{aligned}$$

to find

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{T}} f d\theta &= \int_{\mathbb{T}} \partial_t f d\theta = -\int_{\mathbb{T}} \partial_\theta(\omega[f]f)d\theta\\
&= -(\omega[f]f)(2\pi, \Omega, t) + (\omega[f]f)(0, \Omega, t) = 0.
\end{aligned}$$

This yields the first desired result. By means of it we obtain

$$\frac{d}{dt}\iint_{\mathbb{T}\times\mathbb{R}} f(\theta, \Omega, t)d\Omega d\theta = \int_{\mathbb{R}} \frac{d}{dt}\Big(\int_{\mathbb{T}} f(\theta, \Omega, t)d\theta\Big)d\Omega = 0$$

that gives the second result. □

Remark 8.1.1. *It follows from Lemma 8.1.1 that for any test function $h = h(\Omega)$, we have*

$$\iint_{\mathbb{T}\times\mathbb{R}} h(\Omega) f(\theta,\Omega,t) d\theta d\Omega = \int_{\mathbb{R}} h(\Omega)\Big(\int_{\mathbb{T}} f(\theta,\Omega,t)d\theta\Big) d\Omega$$
$$= \int_{\mathbb{R}} h(\Omega) g(\Omega) d\Omega = \int_{\mathbb{T}\times\mathbb{R}} h(\Omega) f_0(\theta,\Omega)\, d\theta d\Omega.$$

In particular, for $h(\Omega) = \Omega$ and recalling (8.1.1), *we obtain*

$$\iint_{\mathbb{T}\times\mathbb{R}} \Omega f(\theta,\Omega,t) d\theta d\Omega = \int_{\mathbb{R}} \Omega g(\Omega)\, d\Omega = 0\,. \tag{8.1.4}$$

See also (3) in the forthcoming Remark 8.1.2.

A continuity equation for the phase density

In this part, we study the derivation of the transport equation for ρ as follows. We first rewrite the equation (2.3.4) as

$$\partial_t f + \partial_\theta(\Omega f) - K\partial_\theta(L[\rho]f) = 0. \tag{8.1.5}$$

We integrate (8.1.5) with respect to $\Omega \in \mathbb{R}$; by definition of ρ, (8.1.2), we obtain

$$\partial_t \rho + \partial_\theta\left(\int_{\mathbb{R}} \Omega f d\Omega\right) - K\partial_\theta(L[\rho]\rho) = 0.$$

We next present concepts of measure-valued solutions and L^∞-solutions to (2.3.4) following the presentation in [14]. Let $\mathcal{M}(\mathbb{T}\times\mathbb{R})$ be the set of nonnegative Radon measures on $\mathbb{T}\times\mathbb{R} = [0,2\pi]\times\mathbb{R}$, which can be understood as nonnegative bounded linear functionals on $\mathcal{C}(\mathbb{T}\times\mathbb{R})$. For a Radon measure $\nu \in \mathcal{M}(\mathbb{T}\times\mathbb{R})$, we use a standard duality relation:

$$\langle \nu, h\rangle := \int_0^{2\pi}\int_{\mathbb{R}} h(\theta,\Omega)\nu(d\theta,d\Omega), \quad h \in \mathcal{C}_0(\mathbb{T}\times\mathbb{R}).$$

The definition of a measure-valued solution to the equation (2.3.4) is given as follows.

Definition 8.1.1. [14]

1. *For $T \in (0, \infty)$, let $\mu \in L^\infty([0,T); \mathcal{M}(\mathbb{T} \times \mathbb{R}))$ be a measure valued solution to* (2.3.4) *with an initial Radon measure $\mu_0 \in \mathcal{M}(\mathbb{T} \times \mathbb{R})$ if and only if μ satisfies the following conditions:*

   - *μ is weakly continuous:*

   $$\langle \mu_t, h \rangle \quad \textit{is continuous as a function of } t, \quad \forall\, h \in \mathcal{C}_0(\mathbb{T} \times \mathbb{R}).$$

   - *μ satisfies the integral equation: $\forall\, h \in \mathcal{C}_0^1(\mathbb{T} \times \mathbb{R} \times [0,T))$,*

   $$\langle \mu_t, h(\cdot,\cdot,t) \rangle - \langle \mu_0, h(\cdot,\cdot,0) \rangle = \int_0^t \langle \mu_s, \partial_s h + \omega \partial_\theta h \rangle ds, \quad (8.1.6)$$

   where $\omega = \omega(\theta, \Omega, \mu_s)$ is defined by

   $$\omega(\theta, \Omega, \mu_s) := \Omega - K \langle \mu_s, \sin(\theta - \cdot) \rangle. \quad (8.1.7)$$

2. *For $T \in (0, \infty)$, let $f \in L^\infty(\mathbb{T} \times \mathbb{R} \times [0,T))$ be a L^∞ weak solution to* (2.3.4) *with initial data $f_0 \in L^\infty(\mathbb{T} \times \mathbb{R})$ if and only if f satisfies the following conditions:*

   - *The map $t \to f(\cdot,\cdot,t) \in L^1_{loc}(\mathbb{T} \times \mathbb{R})$ is weakly continuous as a function of t.*
   - *f is a distributional solution to* (2.3.4)*: for any test function $\phi \in \mathcal{C}_c^\infty(\mathbb{T} \times \mathbb{R} \times \mathbb{R}_+)$,*

   $$\begin{aligned} &\int_0^\infty \iint_{\mathbb{T}\times\mathbb{R}} (f \partial_t \phi + \omega[f] f \partial_\theta \phi)(\theta, \Omega, t) d\Omega d\theta dt \\ &= \iint_{\mathbb{T}\times\mathbb{R}} f_0(\theta, \Omega) \phi(\theta, \Omega, 0) d\Omega d\theta. \end{aligned}$$

Remark 8.1.2. *1. Let $f = f(\theta, \Omega, t)$ be a classical solution to the kinetic Kuramoto model* (2.3.4)*. Then $\mu_t := f(\theta, \Omega, t) d\Omega d\theta$ is a measure valued solution.*

2. Note that the empirical measure

$$\mu_t = \frac{1}{N} \sum_{i=1}^N \delta_{(\theta_i(t), \Omega_i(t))},$$

is a measure valued solution where $(\theta_i(t), \Omega_i(t))$ *is the solution of*

$$\frac{d\theta_i}{dt} = \Omega_i - \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_i - \theta_j), \qquad \frac{d\Omega_i}{dt} = 0, \quad t > 0.$$

3. Since the distribution function $g(\Omega)$ *has a compact support (see* (8.1.1)*) and the dynamics* (2.3.4) *only governs* θ*-variable, we can see that the projected* Ω*-support of* μ_t *also has a compact support. Hence under the assumption of the compact support of* g*, we can enlarge the class of test functions to* $\mathcal{C}(\mathbb{T} \times \mathbb{R})$*. This justifies also* (8.1.4) *in Remark 8.1.1.*

4. By choosing $h = \Omega$ *in* (8.1.6) *(see above comment in (3)), we have*

$$\langle \mu_t, \Omega \rangle = \langle \mu_0, \Omega \rangle, \quad t > 0.$$

5. It is easy to see that $f_e(\Omega) = \frac{g(\Omega)}{2\pi}$ *is a solution to* (2.3.4)*. Indeed one has*

$$\begin{aligned} L[f_e] &= \int_{\mathbb{T}} \sin(\theta - \theta_*) \left(\int_{\mathbb{R}} f_e(\Omega) d\Omega \right) d\theta_* \\ &= \frac{1}{2\pi}\int_{\mathbb{T}} \sin(\theta - \theta_*) \left(\int_{\mathbb{R}} g(\Omega) d\Omega \right) d\theta_* \\ &= \frac{1}{2\pi}\int_{\mathbb{T}} \sin(\theta - \theta_*) d\theta_* = 0\,, \end{aligned}$$

and hence $\partial_\theta(\omega(f_e)f_e) = \partial_\theta(\Omega f_e(\Omega)) = 0 = \partial_t f_e$*.*

This solution f_e *is called "incoherent solution" which corresponds to the completely "esynchronized" state.*

Lemma 8.1.2. [14] *Suppose that the distribution function* $g = g(\Omega)$ *has a compact support and satisfies*

$$\langle \mu_0, \Omega \rangle = 0,$$

and let $\mu \in L^\infty([0,T); \mathcal{M}(\mathbb{T} \times \mathbb{R}))$ *be a measure valued solution to* (2.3.4) *with finite mass* $||\mu_0|| < \infty$*. Then for* $t \geq 0$*, we have*

$$\langle \mu_t, 1 \rangle = \langle \mu_0, 1 \rangle, \qquad \langle \mu_t, \theta \rangle = \langle \mu_0, \theta \rangle, \qquad t \geq 0.$$

Proof. (i) In (8.1.6), we set $h = 1$. Then the R.H.S. of (8.1.6) will be zero, hence we have conservation of total mass.
(ii) For the time-evolution of the first moment of θ, it follows from Remark 8.1.2 (4) that

$$\langle \mu_t, \Omega \rangle = 0, \quad t > 0.$$

We now set $h(\theta) = \theta$ in (8.1.6) and use (8.1.7) to get

$$\begin{aligned}
\langle \mu_t, \theta \rangle =& \langle \mu_0, \theta \rangle + \int_0^t \langle \mu_s, \omega \rangle ds \\
=& \langle \mu_0, \theta \rangle + \int_0^t \Big(\langle \mu_s, \Omega \rangle - K \langle \mu_s, \langle \mu_s, \sin(\theta - \theta_*) \rangle \rangle \Big) ds \\
=& \langle \mu_0, \theta \rangle - K \int_0^t \langle \mu_s, \langle \mu_s, \sin(\theta - \theta_*) \rangle \rangle ds \\
=& \langle \mu_0, \theta \rangle,
\end{aligned}$$

where we used the anti-symmetry of $\sin(\theta - \theta_*)$ to find

$$\langle \mu_s, \langle \mu_s, \sin(\theta - \theta_*) \rangle \rangle = 0.$$

□

8.2 Global existence of BV weak solutions

In this section, we study the global existence of BV weak solutions to (2.3.4) equation for identical oscillators. Without loss of generality, we assume $g(\Omega) = \delta$ where δ is the Dirac delta located at $\Omega = 0$. As a consequence

$$f(\theta, \Omega, t) = \rho(\theta, t) \otimes \delta(\Omega)$$

and

$$\omega[f]f = (\Omega - KL[\rho])\, \rho \otimes \delta(\Omega) = KL[\rho]\rho\,.$$

Therefore (2.3.4) reduces to an equation for $\rho(\theta, t)$, for which we consider the Cauchy problem:

$$\begin{aligned}
&\partial_t \rho - K \partial_\theta (L[\rho]\rho) = 0. \\
&\rho(\theta, 0) = \rho_0(\theta)
\end{aligned} \tag{8.2.1}$$

with $L[\rho]$ given in (2.3.4), that we recall for convenience:

$$L[\rho](\theta,t) := \int_{\mathbb{T}} \sin(\theta-\theta_*)\rho(\theta_*,t)d\theta_* \,.$$

The initial datum ρ_0 is assumed to belong to the set $\mathcal{X}$ defined by

$$\begin{aligned}\mathcal{X} = \Big\{ &\rho : \mathbb{R}\to\mathbb{R} \text{ } 2\pi\text{-periodic}, \quad \rho \in BV\big(\mathbb{R}/(2\pi\mathbb{Z})\big), \\ &\rho(\theta) \geq 0, \quad \int_{\mathbb{T}} \rho(\theta)\,d\theta = 1 \Big\}.\end{aligned} \tag{8.2.2}$$

8.2.1 Construction of approximate solutions

In this subsection, we present approximate solutions to (8.2.1) that consist of functions which are piecewise constant in space, but not constant in time. The jump discontinuities are located on a finite number of curves.

Suppose that the 2π–periodic initial datum ρ_0 is piecewise constant with a finite number of jump discontinuities. For $N\in\mathbb{N}$, let $\theta_{01}<\theta_{02}<\cdots<\theta_{0N}$ be the location of the discontinuities in the interval $[0,2\pi)$. By periodicity, one has that

$$\rho_0(\theta_{01}-) = \rho_0(\theta_{0N}+).$$

Recalling (8.1.3), we consider a system of characteristic equations:

$$\begin{cases} \dot\theta_i = -KL[\rho](\theta_i), \quad i=1,\ldots,N, \\ \dot\rho_i = K\rho_i \dfrac{L[\rho](\theta_{i+1})-L[\rho](\theta_i)}{\theta_{i+1}-\theta_i} \end{cases} \tag{8.2.3}$$

subject to initial data

$$(\rho_{0i},\theta_{0i}) = (\rho_0(\theta_{0i}+),\theta_{0i}) \tag{8.2.4}$$

where we set $\theta_{N+1}=\theta_1+2\pi$, $\theta_0=\theta_N-2\pi$.

We define our approximate solution ρ by

$$\rho(\theta,t) = \sum_{i=1}^{N} \rho_i(t)\chi_{(\theta_i,\theta_{i+1})}(\theta), \qquad \theta\in\mathbb{T}. \tag{8.2.5}$$

In this case, the operator $L[\rho]$ in (8.2.3) can be rewritten as follows:

$$\begin{aligned} L[\rho(\cdot,t)](\theta) &= \int_{\mathbb{T}} \sin(\theta-\theta_*)\rho(\theta_*,t)\,d\theta_* \\ &= \sum_{k=1}^{N} \rho_k \int_{\theta_k}^{\theta_{k+1}} \sin(\theta-\theta_*)\,d\theta_* \\ &= \sum_{k=1}^{N} \rho_k \left[\cos(\theta-\theta_{k+1}) - \cos(\theta-\theta_k)\right]. \end{aligned} \tag{8.2.6}$$

Note that the system (8.2.3), (8.2.4) and (8.2.6) can be written as an autonomous system in terms of the variables $U = (\theta_1,\ldots,\theta_N,\rho_1,\ldots,\rho_N)$, as $\dot U = F(U)$, with F Lipschitz continuous. Thus, the local existence and uniqueness follow from the standard Cauchy-Lipschitz theory.

8.2.2 Several properties of the approximations

In this subsection, we study the several key properties of the approximate solutions defined in (8.2.5). We next study the conservation of mass for our approximate solutions ρ. We first notice that the relation (8.2.5) implies

$$\|\rho(t)\|_{L^1(\mathbb{T})} = \int_{\mathbb{T}} \rho(\theta,t)d\theta = \sum_{i=1}^{N} \rho_i(t)(\theta_{i+1}-\theta_i(t)).$$

Moreover, we define

$$\mathcal{A}_i(t) := \rho_i(t)(\theta_{i+1}(t)-\theta_i(t)), \qquad \mathcal{A}_{0i} = \mathcal{A}_i(0) = \rho_{0i}\left(\theta_{0(i+1)}-\theta_{0i}\right), \tag{8.2.7}$$

and define I_0 as the L^1 norm of ρ at time $t=0$:

$$I_0 := \|\rho(0)\|_{L^1(\mathbb{T})} = \sum_{i=1}^{N} \rho_{0i}(\theta_{0(i+1)}-\theta_{0i}) = \sum_{i=1}^{N} \mathcal{A}_{0i}. \tag{8.2.8}$$

The L^1 norm I_0 will be assumed to be approximately 1.

The following lemma states the conservation of mass for $t>0$.

Lemma 8.2.1. *Let $(\rho_i(t),\theta_i(t))$ be a solution to the characteristic system* (8.2.3)*. Then, we have following assertions:*

1. *The set of discontinuities is well-ordered in the sense that*

$$\theta_1(t) < \theta_2(t) < \cdots < \theta_N(t), \quad \forall\, t > 0.$$

2. *The total mass is conserved:*

$$\sum_{i=1}^{N} \rho_i(t)(\theta_{i+1}(t) - \theta_i(t)) = \sum_{i=1}^{N} \rho_i(0)(\theta_{i+1}(0) - \theta_i(0)), \quad t \geq 0.$$

More precisely, one has

$$\mathcal{A}_i(t) \equiv \mathcal{A}_{i0} \qquad \forall\, t \geq 0\,.$$

Proof. (1) Recall that the initial jump discontinuities θ_i are well-ordered in the sense that

$$\theta_1(0) < \theta_2(0) < \cdots < \theta_N(0).$$

We claim that the trajectories θ_i do not meet in any finite future time, i.e.,

$$\theta_i(t) < \theta_{i+1}(t), \quad 1 \leq i \leq N,\ t > 0. \tag{8.2.9}$$

Indeed, suppose not, there exists i_0 and t_* such that

$$\theta_{i_0}(t) < \theta_{i_0+1}(t), \quad t < t_* \quad \text{and} \quad \theta_{i_0}(t_*) = \theta_{i_0+1}(t_*).$$

Without loss of generality, we assume that only these two fronts meet at time t_*. Then, we have

$$\begin{aligned} L[\rho](\theta_{i_0}(t_*)) &= \sum_{i=1}^{N} \rho_i(t_*) \left[\cos(\theta_{i_0}(t_*) - \theta_{i+1}(t_*)) - \cos(\theta_{i_0}(t_*) - \theta_i(t_*))\right] \\ &= \sum_{i=1}^{N} \rho_i(t_*) \left[\cos(\theta_{i_0+1}(t_*) - \theta_{i+1}(t_*)) - \cos(\theta_{i_0+1}(t_*) - \theta_i(t_*))\right] \\ &= L[\rho](\theta_{i_0+1}(t_*)). \end{aligned}$$

Consider the system:

$$\dot{\theta}_i = -KL[\rho](\theta_i, t), \quad i \neq i_0, i_0+1, \qquad \dot{\theta}_* = -KL[\rho](\theta_*, t), \quad t > t_*$$

with $\theta_*(t_*) = \theta_{i_0}(t_*) = \theta_{i_0+1}(t_*)$. The function $L[\rho](\theta, t)$ is continuous in θ and t and uniformly Lipschitz continuous in θ. By the uniqueness theorem, the above system with $(N-1)$ variables has a unique solution for $|t-t_*| < \varepsilon$ with $\varepsilon \ll 1$. Since at time $t = t_*$ it has the same data as the system for all $\{\theta_i\}_{i=1}^N$, then the solution with $\theta_{i_0}(t) = \theta_{i_0+1}(t) =: \theta_*(t)$ for $t < t_*$ would be a solution also for the system with $\{\theta_i\}_{i=1}^N$. This contradicts the uniqueness of solutions to system (8.2.3) and completes the proof of the claim (8.2.9).

(2) We use (8.2.7) and (8.2.3) to see

$$\begin{aligned}\dot{\mathcal{A}}_i(t) &= \dot{\rho}_i(t)(\theta_{i+1}(t) - \theta_i(t)) + \rho_i(t)(\dot{\theta}_{i+1}(t) - \dot{\theta}_i(t)) \\ &= K\rho_i \frac{L[\rho](\theta_{i+1}) - L[\rho](\theta_i)}{\theta_{i+1} - \theta_i}(\theta_{i+1}(t) - \theta_i(t)) \\ &\quad - K\rho_i(t)\Big(L[\rho](\theta_{i+1}) - L[\rho](\theta_i)\Big) \\ &= 0.\end{aligned} \tag{8.2.10}$$

This yields the desired identities. □

Now we collect some properties of $L[\rho]$ in the next lemma.

Lemma 8.2.2. *For I_0 as in* (8.2.8)*, the following properties hold.*

(i) $|(\partial_\theta)^k L[\rho(\cdot,t)](\theta)| \le I_0 \quad \forall\, k \ge 0\,, \qquad \mathrm{TV}\, L[\rho(\cdot,t)] \le 2\pi I_0.$

(ii) $|L[\rho(\cdot)](\theta)| = |L\left[\rho(\cdot) - \rho_e\right](\theta)| \le \|\rho(\cdot,t) - \rho_e\|_{L^1(\mathbb{T})}\,, \qquad \rho_e = \dfrac{1}{2\pi}\,.$

More generally, if $\rho_1(\theta,t)$ and $\rho_2(\theta,t)$ are two approximate solutions defined by the algorithm, then

$$|L[\rho_1(\cdot,t) - \rho_2(\cdot,t)](\theta)| \le \|\rho_1(\cdot,t) - \rho_2(\cdot,t)\|_{L^1(\mathbb{T})} \qquad \forall\, \theta \in \mathbb{T}\,. \tag{8.2.11}$$

Proof. Properties *(i)* and *(ii)* follow from straightforward computations. Indeed by (8.2.6) and (8.2.2) one finds that

$$|L[\rho(\cdot,t)](\theta)| = \Big|\int_{\mathbb{T}} \sin(\theta-\theta_*)\rho(\theta_*,t)\,d\theta_*\Big| \le \int_{\mathbb{T}} \rho(\theta_*,t)d\theta_* = I_0,$$

as well as

$$|\partial_\theta L[\rho(\cdot,t)](\theta)| = \Big|\int_{\mathbb{T}} \cos(\theta-\theta_*)\rho(\theta_*,t)\,d\theta_*\Big| \le \int_{\mathbb{T}} \rho(\theta_*,t)d\theta_* = I_0. \tag{8.2.12}$$

The same argument clearly holds for any higher derivative $(\partial_\theta)^k L$. Moreover,

$$\mathrm{TV}\, L[\rho(\cdot,t)] = \int_{\mathbb{T}} |\partial_\theta L[\rho(\cdot)](\theta)|\, d\theta \le \int_{\mathbb{T}} d\theta = 2\pi I_0\,,$$

and

$$|L[\rho(\cdot)](\theta)| = \Big| \int_{\mathbb{T}} \sin(\theta-\theta_*)\, [\rho(\theta_*,t)-\rho_e]\; d\theta_* \Big| \le \int_{\mathbb{T}} |\rho(\theta_*,t)-\rho_e|\; d\theta_*\,.$$

Notice that *(ii)* is a special case of (8.2.11), since $\rho_{i0}=\rho_e$ for every index i leads to $\rho(\theta,t)\equiv\rho_e$.

The proof of (8.2.11) is completely analogous to the one of this special case. □

Remark 8.2.1. *As a consequence of Lemma 8.2.2, we find that for* $\forall\ k\ge 0\,,\ t>0\,,\ \theta\in\mathbb{T}\,,$

$$\begin{aligned} &|(\partial_\theta)^k L[\rho(\cdot,t)](\theta)| \le \min\{I_0, \|\rho(\cdot,t)-\rho_e\|_{L^1(\mathbb{T})}\} =: b(t), \\ &\mathrm{TV}\, L[\rho(\cdot,t)] \le 2\pi\cdot b(t). \end{aligned} \tag{8.2.13}$$

We next study the boundedness of ρ and of $\mathrm{TV}\,\rho$ in the following lemma.

Lemma 8.2.3. *Let* $(\rho_i(t),\theta_i(t))$ *be a solution to the characteristic system* (8.2.3)*. Then, we have following estimates:*

$$\begin{aligned} &(i) \quad \rho_i(0)\mathrm{e}^{-Kt} \le \rho_i(t) \le \rho_i(0)\mathrm{e}^{Kt}. \\ &(ii) \quad \mathrm{TV}\,\rho(\cdot,t) \le \mathrm{e}^{KI_0t}\cdot \mathrm{TV}\,\rho(\cdot,0) \;+\; 2\Big[\mathrm{e}^{KI_0t}-1\Big]. \end{aligned}$$

Proof. *(i)* Notice that ρ_i satisfies

$$\dot\rho_i = K\rho_i \frac{L[\rho](\theta_{i+1})-L[\rho](\theta_i)}{\theta_{i+1}-\theta_i} = K\rho_i(\partial_\theta L[\rho])(\theta_i^*). \tag{8.2.14}$$

where θ_i^* is a value between θ_i and θ_{i+1}. On the other hand, we have

$$|\partial_\theta L[\rho(\cdot)](\theta)| = \Big| \int_{\mathbb{T}} \cos(\theta-\theta_*)\rho(\theta_*,t)\, d\theta_* \Big| \le \int_{\mathbb{T}} \rho(\theta_*,t)d\theta_* = 1. \tag{8.2.15}$$

Thus, (8.2.14) and (8.2.15) imply

$$-K\rho_i \le \dot\rho_i \le K\rho_i.$$

This yields the desired estimates.

(ii) Observe that the total variation of ρ is given by:

$$\mathrm{TV}\,\rho(\cdot,t) = \sum_{i=1}^{N} |\rho_{i+1} - \rho_i|, \quad \rho_{N+1} = \rho_1,$$

and then

$$\begin{aligned}
\frac{d}{dt}\mathrm{TV}\,\rho(\cdot,t) &= \sum_{i=1}^{N} \mathrm{sgn}(\rho_{i+1} - \rho_i)\,(\dot\rho_{i+1} - \dot\rho_i) \\
&= K\sum_{i=1}^{N} \mathrm{sgn}(\rho_{i+1} - \rho_i) \\
&\qquad \times \left(\rho_{i+1}\cdot\frac{\Delta L}{\Delta\theta}(\theta_{i+2},\theta_{i+1}) - \rho_i\cdot\frac{\Delta L}{\Delta\theta}(\theta_{i+1},\theta_i)\right) \qquad (8.2.16)\\
&=: K\sum_{i=1}^{N} \mathrm{sgn}(\rho_{i+1} - \rho_i)\cdot \mathcal{J}_i \qquad (8.2.17)
\end{aligned}$$

where we used the notation

$$\frac{\Delta L}{\Delta\theta}(\theta_{i+1},\theta_i) = \frac{L[\rho](\theta_{i+1}) - L[\rho](\theta_i)}{\theta_{i+1}-\theta_i}, \quad i = 1,\ldots,N\,.$$

Now we compute the quantity $\mathcal{J}_i$ in (8.2.17):

$$\begin{aligned}
\mathcal{J}_i &= (\rho_{i+1}-\rho_i)\,\partial_\theta L[\rho](\theta_{i+1}) + \rho_{i+1}\cdot\left[\frac{\Delta L}{\Delta\theta}(\theta_{i+2},\theta_{i+1}) - \partial_\theta L[\rho](\theta_{i+1})\right] \\
&\quad - \rho_i\cdot\left[\frac{\Delta L}{\Delta\theta}(\theta_{i+1},\theta_i) - \partial_\theta L[\rho](\theta_{i+1})\right] \\
&= (\rho_{i+1}-\rho_i)\,\partial_\theta L[\rho](\theta_{i+1}) + \rho_{i+1}\cdot\partial^2_{\theta\theta}L[\rho](\theta^{**}_{i+1})(\theta^*_{i+1}-\theta_{i+1}) \\
&\quad - \rho_i\cdot\partial^2_{\theta\theta}L[\rho](\theta^{**}_i)(\theta^*_i - \theta_{i+1})
\end{aligned}$$

for some θ^*_{i+1}, $\theta^{**}_{i+1}\in(\theta_{i+1},\theta_{i+2})$ and θ^*_i, $\theta^{**}_i\in(\theta_i,\theta_{i+1})$.

Recalling the estimates (8.2.13) and the definition of $b(t)$, we find that

$$\frac{d}{dt}\mathrm{TV}\,\rho(\cdot,t) \le K\sum_{i=1}^{N}|\rho_{i+1}-\rho_i|\,\partial_\theta L[\rho](\theta_{i+1})$$

$$
\begin{aligned}
&+ K \sum_{i=1}^{N} b(t) \left[\rho_{i+1}(\theta_{i+2} - \theta_{i+1}) + \rho_i(\theta_{i+1} - \theta_i)\right] \\
&\leq K \cdot b(t) \cdot (\mathrm{TV}\, \rho(\cdot, t) + 2) \qquad (8.2.18) \\
&\leq K \cdot I_0 \cdot (\mathrm{TV}\, \rho(\cdot, t) + 2)\ .
\end{aligned}
$$

As a consequence, the following inequality for $\mathrm{TV}\, \rho(\cdot, t)$ holds:

$$\mathrm{TV}\, \rho(\cdot, t) \leq \mathrm{e}^{K I_0 t} \Big[\mathrm{TV}\, \rho(\cdot, 0) + 2 \Big] - 2\,.$$

The proof of *(ii)* is complete.

□

8.2.3 A global existence of weak solution

In this subsection, we will provide a global existence of weak solutions, that are defined as follows.

Definition 8.2.1. *Let $T > 0$. A function $\rho : \mathbb{T} \times [0, T] \mapsto [0, \infty)$ is an entropy weak solution to* (8.2.1) *with initial data $\rho_0 \in L^\infty(\mathbb{T})$ if the following holds.*

(1) *The map $[0, T] \ni t \mapsto \rho(\cdot, t) \in L^\infty(\mathbb{T})$ is continuous, with $\rho(\cdot, 0) = \rho_0$.*

(2) *$\rho(\theta, t)$ satisfies, for all $\alpha \in \mathbb{R}$*

$$\partial_t \left|\rho - \alpha\right| - K \partial_\theta \left[\ell(\theta, t) \left|\rho - \alpha\right|\right] - \mathrm{sgn}(\rho - \alpha) K \alpha (\partial_\theta \ell) = 0 \qquad (8.2.19)$$

in the sense of distributions, with

$$\ell(\theta, t) = L[\rho(\cdot, t)](\theta).$$

Remark 8.2.2. *(i) In the assumption* **(1)** *above on ρ, the integral term $\ell(\theta, t)$ is well defined, continuous in (θ, t) and then bounded on $\mathbb{T} \times [0, T]$, as well as $\partial_\theta \ell(\theta, t)$.*

(ii) By setting $\alpha = 0$ in (8.2.19)*, it follows that ρ is a distributional solution of the scalar balance law (with periodic boundary conditions)*

$$\begin{cases} \partial_t \rho - K \partial_\theta (\ell(\theta, t) \rho) = 0, \\ \rho(\theta, 0) = \rho_0(\theta). \end{cases}$$

(iii) For $\alpha = \rho_e = \frac{1}{2\pi}$ in (8.2.19), then one has

$$\begin{aligned}
&\int_{\mathbb{T}} |\rho(\theta,t) - \rho_e| \, d\theta \\
&= \int_{\mathbb{T}} |\rho_0(\theta) - \rho_e| \, d\theta + K\rho_e \int_0^T \int_{\mathbb{T}} \operatorname{sgn}(\rho - \rho_e) \partial_\theta L[\rho(\cdot,t)](\theta) \, d\theta \, .
\end{aligned}$$

On the other hand, by triangular inequality one finds that

$$\int_{\mathbb{T}} |\rho(\theta,t) - \rho_e| \, d\theta \le \int_{\mathbb{T}} [\rho(\theta,t) + \rho_e] \, d\theta = 2$$

The existence of BV solutions is established in the following theorem.

Theorem 8.2.1. *Let $T > 0$ and the initial data $\rho_0 \in \mathcal{X}$, defined in* (8.2.2). *Then the Cauchy problem* (8.2.1) *admits a entropy weak solution $\rho = \rho(\theta,t)$ in the sense of Definition 8.2.1. Moreover $\rho(\cdot,t) \in \mathcal{X}$ for every $t \in [0,T]$.*

Proof. Let $\rho_0 \in \mathcal{X}$ and $\varepsilon > 0$. We choose the initial approximation ρ_{0i} and θ_{0i}, $i = 1, \ldots, N = N(\varepsilon)$, given in Subsection 8.2.1, such that

$$\begin{aligned}
&\max_{1\le i\le N} |\theta_{i+1}(0) - \theta_i(0)| < \varepsilon \, , \qquad \rho_{0i} = \rho_0(\theta_i+) \, , \\
&\max_{1\le i\le N} |\mathcal{A}_i(0)| = \max_{1\le i\le N} \rho_i(0) \, |\theta_{i+1}(0) - \theta_i(0)| < \varepsilon \, .
\end{aligned}$$

Let ρ^ε be the approximate solution given in (8.2.3)–(8.2.5) and define $I_{0,\varepsilon}$ as its L^1-norm on $\mathbb{T}$ at time $t = 0$, as in (8.2.8). One has the immediate relation $\mathrm{TV}\, \rho^\varepsilon(\cdot,0) \le \mathrm{TV}\, \rho_0$, that

$$|I_{0,\varepsilon} - 1| \ \le \ \varepsilon \, \mathrm{TV} \{\rho_0; \mathbb{T}\} \tag{8.2.20}$$

and hence that $\|\rho_0 - \rho^\varepsilon(\cdot,0)\|_{L^1(\mathbb{T})} = |I_{0,\varepsilon} - 1| \to 0$ as $\varepsilon \to 0$.

Indeed, to prove (8.2.20), we observe that

$$\begin{aligned}
\left| \int_{\mathbb{T}} [\rho_0(\theta) - \rho^\varepsilon(\theta,0)] \, d\theta \right| \ &\le \ \sum_{i=1}^N \int_{\theta_i}^{\theta_{i+1}} |\rho_0(\theta) - \rho_0(\theta_i+)| \, d\theta \\
&\le \ \sum_{i=1}^N \int_{\theta_i}^{\theta_{i+1}} \mathrm{TV} \{\rho_0; (\theta_i, \theta_{i+1})\} \, d\theta
\end{aligned}$$

$$\leq \varepsilon \, \mathrm{TV}\{\rho_0; \mathbb{T}\}.$$

Thanks to the bounds provided in Lemma 8.2.3, there exists a subsequence $\varepsilon_j \to 0$ and limit functions $\rho(\theta,t)$, $\tilde{L}(\theta,t)$ such that

$$\rho^{\varepsilon_j} \to \rho, \quad L[\rho^{\varepsilon_j}] \to \tilde{L} \qquad \text{in } L^1(\mathbb{T}\times[0,T]).$$

Moreover $\rho(\cdot,0) = \rho_0$ and

$$I_{0,\varepsilon} = \int_{\mathbb{T}} \rho^{\varepsilon}(\theta,t)\,d\theta \to \int_{\mathbb{T}} \rho(\theta,t)\,d\theta$$

so that $\int_{\mathbb{T}} \rho(\theta,t)\,d\theta = 1$ for every $t > 0$. Also, recalling the definition of L in (8.2.6), one has that

$$\tilde{L}(\theta,t) = \int_{\mathbb{T}} \sin(\theta-\theta_*)\rho(\theta_*,t)\,d\theta_*.$$

Next we prove that ρ satisfies **(2)** in Definition 8.2.1. To prove (8.2.19), let $\alpha \in \mathbb{R}$ and $\phi = \phi(\theta,t) \in \mathcal{C}^1(\mathbb{T}\times(0,T))$ a test function. Set

$$\begin{aligned} I^{\varepsilon}(T) := \int_0^T \int_{\mathbb{T}} \Big(&- |\rho^{\varepsilon}-\alpha|\,\partial_t\phi + KL[\rho^{\varepsilon}(\cdot,t)](\theta)\,|\rho^{\varepsilon}-\alpha|\,\partial_\theta\phi \\ &- \mathrm{sgn}(\rho^{\varepsilon}-\alpha)K\partial_\theta(L[\rho^{\varepsilon}])\alpha\phi\Big)d\theta dt. \end{aligned}$$

By possibly taking a further subsequence $\varepsilon_j \to 0$, one has that

$$\begin{aligned} I^{\varepsilon_j}(T) \to \int_0^T \int_{\mathbb{T}} \Big(&- |\rho-\alpha|\,\partial_t\phi + KL[\rho(\cdot,t)](\theta)\,|\rho-\alpha|\,\partial_\theta\phi \\ &- \mathrm{sgn}(\rho-\alpha)K\partial_\theta(L[\rho])\alpha\phi\Big)d\theta dt. \end{aligned}$$

Now we are going to prove that $I^{\varepsilon} \to 0$ as $\varepsilon \to 0$.

Indeed, after integration by parts, one has

$$\begin{aligned} I^{\varepsilon}(T) = \int_0^T \int_{\mathbb{T}} \Big[&\partial_t\,|\rho^{\varepsilon}-\alpha| - K\partial_\theta\left(L[\rho^{\varepsilon}]\,|\rho^{\varepsilon}-\alpha|\right) \\ &- \mathrm{sgn}(\rho^{\varepsilon}-\alpha)K(\partial_\theta L[\rho^{\varepsilon}])\alpha\Big]\phi(\theta,t)\,d\theta dt \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{N} \int_0^T \int_{\theta_i}^{\theta_{i+1}} \operatorname{sgn}(\rho_i - \alpha) \\
&\quad \times \Big[\dot{\rho}_i - K(\rho_i - \alpha)(\partial_\theta L[\rho^\varepsilon]) - K\partial_\theta(L[\rho^\varepsilon])\alpha\Big]\phi(\theta,t)\, d\theta dt \\
&= \sum_{i=1}^{N} \int_0^T \int_{\theta_i}^{\theta_{i+1}} \operatorname{sgn}(\rho_i - \alpha)\Big[\dot{\rho}_i - K\rho_i\,\partial_\theta(L[\rho^\varepsilon])\Big]\phi(\theta,t)\, d\theta dt.
\end{aligned}
$$

Dropping ε for simplicity, it follows from (8.2.3) that

$$
\begin{aligned}
\dot{\rho}_i - K\rho_i\partial_\theta L[\rho](\theta) &= K\rho_i\left[\frac{L[\rho](\theta_{i+1}) - L[\rho](\theta_i)}{\theta_{i+1}-\theta_i} - \partial_\theta L[\rho](\theta)\right] \\
&= K\rho_i\left[\partial_\theta L[\rho](\xi_i) - \partial_\theta L[\rho](\theta)\right] \\
&= K\rho_i(\partial_\theta^2 L[\rho])(\eta_i)|\,(\xi_i - \theta)
\end{aligned}
$$

for some ξ_i, $\eta_i \in (\theta_i, \theta_{i+1})$.

Therefore, using (8.2.20), we find that

$$
\begin{aligned}
|I^\varepsilon(T)| &\le K\|\phi\|_{L^\infty}\underbrace{\max_{\theta\in\mathbb{T}}|\partial_\theta^2 L[\rho](\theta)|}_{\le I_{0,\varepsilon}}\sum_{i=1}^{N}\int_0^T \rho_i\int_{\theta_i}^{\theta_{i+1}}|\xi_i-\theta|\; d\theta dt \\
&\le \frac{K}{2}\|\phi\|_{L^\infty}(1+\varepsilon\,\mathrm{TV}\{\rho_0\})\sum_{i=1}^{N}\int_0^T \rho_i\,|\theta_{i+1}-\theta_i|^2\; dt \\
&\le \frac{K}{2}\|\phi\|_{L^\infty}(1+\varepsilon\,\mathrm{TV}\{\rho_0\})\left[\max_{1\le i\le N}\mathcal{A}_{i0}\right]\sum_{i=1}^{N}\int_0^T |\theta_{i+1}-\theta_i|\; dt \\
&\le K\|\phi\|_{L^\infty}(1+\varepsilon\,\mathrm{TV}\{\rho_0\})\,\varepsilon\,\pi T\,.
\end{aligned}
$$

This yields the existence part of the desired result. □

8.3 Exponentially growing solution

In this section, we provide an exponentially growing mode solution which is a perturbation of the incoherent solution $\rho_e = \frac{1}{2\pi}$, which exhibits the instability of the incoherent solution $f_e = \frac{g(\Omega)}{2\pi}$. For relevant physics literature, we refer to [56, 58].

Consider a Cauchy problem to the Kuramoto-Sakaguchi equation for phase density ρ:

$$\begin{aligned} &\partial_t \rho - K\partial_\theta(L[\rho]\rho) = 0. \\ &\rho(\theta, 0) = \rho_e + \hat{\rho}_0. \end{aligned}$$

Later, we will show that there exists a perturbed solution ρ satisfying

$$|\rho(t)| \geq Ce^{-\Lambda t}, \quad \text{as } t \to \infty, \quad \text{for some positive constants } C, \Lambda > 0.$$

8.3.1 Construction of an exponentially growing mode

In this subsection, we provide an existence of an exponentially growing $L^\infty \cap$ BV-solution from some initial datum ρ_0, which corresponds to the perturbation of the incoherent solution ρ_e. This results in the nonlinear instability of the incoherent of the Kuramoto-Sakaguchi equation for identical oscillators.

As an initial datum ρ_0, we will consider a symmetric function which has two stage discontinuities (see Figure 8.1):

$$\rho_0(\theta) = \begin{cases} \bar{\rho}_1 & -\theta^* < \theta < \theta^*, \\ \bar{\rho}_2 & \theta \in (-\pi, -\theta^*) \cup (\theta^*, \pi)\,, \end{cases} \tag{8.3.1}$$

where $\bar{\rho}_i$ and θ^* satisfy the unit mass condition:

$$0 < \theta^* < \pi, \quad \bar{\rho}_1 > \frac{1}{2\pi} > \bar{\rho}_2 > 0\,, \qquad \bar{\rho}_1\theta^* + \bar{\rho}_2(\pi - \theta^*) = \frac{1}{2}\,. \tag{8.3.2}$$

From now on, we proceed to construct an exponentially growing solution with the well-prepared initial data (8.3.1) - (8.3.2).

For a positive constant $N \geq 2$, we set $\Delta\theta := \frac{\pi}{N}$ and divide the interval $(-\pi, \pi)$ in to $2N$-equal subintervals, which have same length $\Delta\theta$, and define the initial configurations of θ_i^N and ρ_i^N to satisfy

$$
\begin{aligned}
&\theta_i^N(0) := i\Delta\theta = \frac{i\pi}{N}, \qquad i = -N, ..., N,\\
&\rho^N(0) := \sum_{\ell=1}^{N} \rho_i^N(0)\chi_{(\theta_{i-1}^N(0),\theta_i^N(0))} + \sum_{i=-N}^{-1} \rho_i^N(0)\chi_{(\theta_i^N(0),\theta_{i+1}^N(0))},\\
&\rho_i^N(0) := \begin{cases} \bar{\rho}_1 & 1 \le |i| \le i_0,\\ \bar{\rho}_2 & i_0 < |i| \le N \end{cases}
\end{aligned}
\tag{8.3.3}
$$

where $i_0 \in \{1, \dots, N-1\}$ an index satisfying

$$(i_0 - 1)\Delta\theta < \theta^* \le i_0\Delta\theta.$$

Note that as long as there is no confusion, we relabeled ρ_i in a symmetric way, $\rho_{-N}, \cdots, \rho_{-1}, \rho_1, \cdots, \rho_N$. We set

$$
\mathcal{A}_i^N := \begin{cases} \rho_i^N(0)\left(\theta_i^N(0) - \theta_{i-1}^N(0)\right) & 1 \le i \le N\\ \rho_i^N(0)\left(\theta_{i+1}^N(0) - \theta_i^N(0)\right) & -N \le i \le -1, \end{cases}
$$

$$
\text{i.e.,} \quad \mathcal{A}_i^N = \begin{cases} \bar{\rho}_1\Delta\theta & 1 \le |i| \le i_0\\ \bar{\rho}_2\Delta\theta & i_0 < |i| \le N \end{cases}
$$

We use (8.3.2) to see that

$$
\begin{aligned}
\sum_{i=1}^{N} \mathcal{A}_i^N &= \bar{\rho}_1 i_0\Delta\theta + \bar{\rho}_2(N - i_0)\Delta\theta = \bar{\rho}_1 i_0\Delta\theta + \bar{\rho}_2(\pi - i_0\Delta\theta)\\
&= (\bar{\rho}_1 - \bar{\rho}_2)\theta^* + \bar{\rho}_2\pi + (\bar{\rho}_1 - \bar{\rho}_2)(i_0\Delta\theta - \theta^*)\\
&= \frac{1}{2} + (\bar{\rho}_1 - \bar{\rho}_2)\left(i_0\Delta\theta - \theta^*\right).
\end{aligned}
$$

This yields

$$
\sum_{i=1}^{N} \mathcal{A}_i^N \ge \frac{1}{2} \quad \text{and} \quad \left|\sum_{i=1}^{N} \mathcal{A}_i^N - \frac{1}{2}\right| \le (\bar{\rho}_1 - \bar{\rho}_2)\,\Delta\theta\,. \tag{8.3.4}
$$

Then, we solve the following ODE system:

$$\begin{cases} \dot{\theta}_i^N = -KL[\rho^N](\theta_i^N), \quad i = -N, ..., N-1, \\ \dot{\rho}_i^N = K\rho_i^N \dfrac{L[\rho](\theta_{i+1}^N) - L[\rho^N](\theta_i^N)}{\theta_{i+1}^N - \theta_i^N}, \end{cases} \tag{8.3.5}$$

subject to initial data

$$(\rho_{i0}^N, \theta_{i0}^N) = (\rho_i^N(0), \theta_i^N(0)).$$

We now define an approximate solution ρ^N:

$$\rho^N(t) := \sum_{i=1}^{N} \rho_i^N(t)\chi_{(\theta_{i-1}^N(t), \theta_i^N(t))} + \sum_{i=-N}^{-1} \rho_i^N(t)\chi_{(\theta_i^N(t), \theta_{i+1}^N(t))}.$$

From now on, as long as there is no confusion, we suppress N-dependence in ρ^N, ρ_i^N and θ_i^N, i.e.,

$$\rho := \rho^N, \quad \rho_i := \rho_i^N, \quad \theta_i := \theta_i^N.$$

8.3.2 Dynamics of approximate solutions

In this subsection, we study the dynamics of θ_i and ρ_i. We next argue that due to the symmetry of initial data, it suffices to consider only for $i = 1, \cdots, N-1$.

Lemma 8.3.1. *Let (θ_i, ρ_i) be a solution to (8.2.3). Then, it satisfies a parity condition:*

$$\rho_i(t) = \rho_{-i}(t), \quad \theta_i(t) = -\theta_{-i}(t), \quad 0 \le i \le N, \ t \ge 0.$$

Proof. Due to the symmetry of initial data and mean-field nature of system (8.2.3), the result is clearly expected and we present its proof below.

• Step A: We introduce new dependent variables:

$$\tilde{\theta}_i = -\theta_{-i}, \quad |i| \le N \quad \text{and} \quad \tilde{\rho}_i = \rho_{-i}, \quad |i| \le N.$$

We next show that $\tilde{\theta}_i$ and $\tilde{\rho}_i$ satisfy the same system (8.2.3):

$$\begin{cases} \dot{\tilde{\theta}}_i = -KL[\tilde{\rho}](\tilde{\theta}_i), \quad |i| \le N, \\ \dot{\tilde{\rho}}_i = K\tilde{\rho}_i \dfrac{L[\tilde{\rho}](\tilde{\theta}_{i+1}) - L[\tilde{\rho}](\tilde{\theta}_i)}{\tilde{\theta}_{i+1} - \tilde{\theta}_i}, \end{cases} \tag{8.3.6}$$

$\diamond$ (Derivation of $(8.3.6)_1$): We first recall that for $i = -N, \cdots, N$,

$$\begin{aligned} L[\rho](\theta_i) = &\sum_{k=-N}^{-1} \rho_k \left[\cos(\theta_i - \theta_{k+1}) - \cos(\theta_i - \theta_k)\right] \\ &+ \sum_{k=1}^{N} \rho_k \left[\cos(\theta_i - \theta_k) - \cos(\theta_i - \theta_{k-1})\right]. \end{aligned}$$

We claim:

$$L[\rho](\theta_{-i}) = -L[\tilde{\rho}](\tilde{\theta}_i), \quad |i| \le N. \tag{8.3.7}$$

Proof of (8.3.7): We use change of variable $k' = -k$ to obtain

$$\begin{aligned} L[\rho](\theta_{-i}) = &\sum_{k=-N}^{-1} \rho_k \left[\cos(\theta_{-i} - \theta_{k+1}) - \cos(\theta_{-i} - \theta_k)\right] \\ &+ \sum_{k=1}^{N} \rho_k \left[\cos(\theta_{-i} - \theta_k) - \cos(\theta_{-i} - \theta_{k-1})\right] \\ = &\sum_{k=-N}^{-1} \tilde{\rho}_{-k} \left[\cos(\tilde{\theta}_i - \tilde{\theta}_{-k-1}) - \cos(\tilde{\theta}_i - \tilde{\theta}_{-k})\right] \\ &+ \sum_{k=1}^{N} \tilde{\rho}_{-k} \left[\cos(\tilde{\theta}_i - \tilde{\theta}_{-k}) - \cos(\tilde{\theta}_i - \tilde{\theta}_{-k+1})\right] \\ = &\sum_{k'=1}^{N} \tilde{\rho}_{k'} \left[\cos(\tilde{\theta}_i - \tilde{\theta}_{k'-1}) - \cos(\tilde{\theta}_i - \tilde{\theta}_{k'})\right] \\ &+ \sum_{k'=-N}^{-1} \tilde{\rho}_{k'} \left[\cos(\tilde{\theta}_i - \tilde{\theta}_{k'}) - \cos(\tilde{\theta}_i - \tilde{\theta}_{k'+1})\right] \\ = &- \sum_{k'=-N}^{-1} \tilde{\rho}'_k \left[\cos(\tilde{\theta}_i - \tilde{\theta}_{k'+1}) - \cos(\tilde{\theta}_i - \tilde{\theta}_{k'})\right] \end{aligned}$$

$$
\begin{aligned}
&- \sum_{k'=1}^{N} \tilde{\rho}_{k'} \left[\cos(\tilde{\theta}_i - \tilde{\theta}_{k'}) - \cos(\tilde{\theta}_i - \tilde{\theta}_{k'-1}) \right] \\
&= -L[\tilde{\rho}](\tilde{\theta}_i).
\end{aligned}
$$

We now use (8.3.7) to find

$$
\dot{\tilde{\theta}}_i = -\dot{\theta}_{-i} = KL[\rho](\theta_{-i}) = -KL[\tilde{\rho}](\tilde{\theta}_i). \tag{8.3.8}
$$

$\diamond$ (Derivation of $(8.3.6)_2$): Similarly, we again convert all dependent variables ρ and θ into $\hat{\rho}$ and $\tilde{\theta}$ variables and use (8.3.7):

$$
L[\rho](\theta_{-i+1}) = L[\rho](\theta_{-(i-1)}) = -L[\tilde{\rho}](\tilde{\theta}_{i-1}), \quad L[\rho](\theta_{-i}) = -L[\tilde{\rho}](\tilde{\theta}_i),
$$

to obtain

$$
\dot{\tilde{\rho}}_i = K\rho_{-i} \frac{L[\rho](\theta_{-i+1}) - L[\rho](\theta_{-i})}{\theta_{-(i-1)} - \theta_{-i}} = K\tilde{\rho}_i \frac{L[\tilde{\rho}](\tilde{\theta}_i) - L[\tilde{\rho}](\tilde{\theta}_{i-1})}{\tilde{\theta}_i - \tilde{\theta}_{i-1}}. \tag{8.3.9}
$$

Finally we combine (8.3.8) and (8.3.9) to see that converted variables $(\tilde{\rho}, \tilde{\theta})$ satisfies the same system as (ρ, θ).

• Step B: Since (ρ_i, θ_i) and $(\tilde{\rho}_i, \tilde{\theta}_i)$ satisfy the same system and initial data, by the uniqueness of the solution to ODE system with Lipschitz vector field, we have

$$
\rho_i(t) = \rho_{-i}(t), \quad \theta_i(t) = -\theta_{-i}(t), \qquad 1 \le i \le N.
$$

□

Remark 8.3.1. *As a direct corollary of Lemma 8.3.1, we have*

$$
(\theta_0, \theta_N)(t) = (0, \pi), \quad t \ge 0
$$

.

In the sequel, the mean-field functional $\mathcal{S}$:

$$
\mathcal{S}(t) := 2\sum_{i=1}^{N} \rho_i (\sin\theta_i - \sin\theta_{i-1}) \tag{8.3.10}
$$

will appear naturally. So we present some equivalent forms for $\mathcal{S}$ in the following lemma.

Lemma 8.3.2. *The functional* $\mathcal{S}$ *defined in* (8.3.10) *can have several equivalent representations:*

$$\mathcal{S} = 2\sum_{i=1}^{N-1}(\rho_i - \rho_{i+1})\sin\theta_i = 2\sum_{i=1}^{N}\mathcal{A}_i\cos\xi_i,$$

where $\xi_i \in \big(\theta_{i-1}(t), \theta_i(t)\big)$ *is a solution of the following trigonometric equation:*

$$\cos\xi_i = \frac{\sin\theta_i - \sin\theta_{i-1}}{\theta_i - \theta_{i-1}}, \quad i = 1, \cdots, N-1. \tag{8.3.11}$$

Proof. • Case A (the first relation): We use Lemma 8.3.1 and (8.3.10) to find

$$\begin{aligned}
\mathcal{S} &= 2\sum_{i=1}^{N}\rho_i(\sin\theta_i - \sin\theta_{i-1})\\
&= 2\rho_1(\sin\theta_1 - \sin\theta_0) + 2\rho_2(\sin\theta_2 - \sin\theta_1) + \cdots + 2\rho_N(\sin\theta_N - \sin\theta_{N-1})\\
&= -2\rho_1\sin\theta_0 + 2(\rho_1 - \rho_2)\sin\theta_1 + \cdots + 2(\rho_{N-1} - \rho_N)\sin\theta_{N-1} + 2\rho_N\sin\theta_N\\
&= 2\sum_{i=1}^{N-1}(\rho_i - \rho_{i+1})\sin\theta_i,
\end{aligned}$$

where we used the fact $\theta_0(t) = 0,\ \theta_N(t) = \pi, \quad t \geq 0$.

• Case B (the second relation): We set $\xi_i = \xi_i(t)$ in the interval $\big(\theta_{i-1}(t), \theta_i(t)\big)$ as a solution of the following equation:

$$\begin{aligned}
&\cos\xi_i = \frac{\sin\theta_i - \sin\theta_{i-1}}{\theta_i - \theta_{i-1}}, \quad \text{or equivalently}\\
&\sin\theta_i - \sin\theta_{i-1} = (\cos\xi_i)(\theta_i - \theta_{i-1}), \quad i = 1, \cdots, N.
\end{aligned} \tag{8.3.12}$$

Then, we use (8.3.10) and (8.3.12) to find the second relation:

$$\mathcal{S}(t) = 2\sum_{i=1}^{N}\rho_i(\theta_i - \theta_{i-1})\cos\xi_i = 2\sum_{i=1}^{N}\mathcal{A}_i\cos\xi_i$$

□

Proposition 8.3.1. *The functions θ_i and ρ_i satisfies Adler's equation:*

$$\begin{cases} \dot{\theta}_i = -K\mathcal{S}\sin\theta_i, \quad i = -N, \cdots, N, \\ \dot{\rho}_i = K(\cos\xi_i)\mathcal{S}\rho_i, \end{cases} \tag{8.3.13}$$

where $\mathcal{S}$ and ξ_i are a mean-field term and the value defined in (8.3.10) *and* (8.3.11)*, respectively.*

Proof. Below, we will derive the equation for θ_i and ρ_i separately.
• Case A (Derivation of equation for θ_i): Recall that

$$\begin{aligned} \dot{\theta}_i = &-K\sum_{k=-N}^{-1} \rho_k \left[\cos(\theta_i - \theta_{k+1}) - \cos(\theta_i - \theta_k)\right] \\ &- K\sum_{k=1}^{N} \rho_k \left[\cos(\theta_i - \theta_k) - \cos(\theta_i - \theta_{k-1})\right]. \end{aligned} \tag{8.3.14}$$

We use Lemma 8.3.1 to find

$$\begin{aligned} \dot{\theta}_i &= -K\sum_{k=1}^{N} \rho_k \left[\cos(\theta_i + \theta_{k-1}) - \cos(\theta_i + \theta_k)\right] \\ &\quad - K\sum_{k=1}^{N} \rho_k \left[\cos(\theta_i - \theta_k) - \cos(\theta_i - \theta_{k-1})\right] \\ &= -K\sum_{k=1}^{N} \rho_k \left[\cos(\theta_i + \theta_{k-1}) - \cos(\theta_i + \theta_k) + \cos(\theta_i - \theta_k) - \cos(\theta_i - \theta_{k-1})\right] \\ &= -K\sum_{k=1}^{N} \rho_k \left[\cos(\theta_i + \theta_{k-1}) - \cos(\theta_i - \theta_{k-1}) + \cos(\theta_i - \theta_k) - \cos(\theta_i + \theta_k)\right] \\ &= -2K\sin(\theta_i)\sum_{k=1}^{N} \rho_k \left[\sin(\theta_k) - \sin(\theta_{k-1})\right] \\ &= -K\mathcal{S}\sin\theta_i. \end{aligned}$$

• Case B (Derivation of equation for ρ_i): It follows from (8.2.10) that

$$\dot{A}_i = \dot{\rho}_i(\theta_i - \theta_{i-1}) + \rho_i(\dot{\theta}_i - \dot{\theta}_{i-1}) = 0.$$

This yields

$$\dot{\rho}_i = -\rho_i \frac{\dot{\theta}_i - \dot{\theta}_{i-1}}{\theta_i - \theta_{i-1}}. \tag{8.3.15}$$

On the other hand, note that

$$\dot{\theta}_i - \dot{\theta}_{i-1} = -K(\sin\theta_i - \sin\theta_{i-1})\mathcal{S}(t) = -K(\theta_i - \theta_{i-1})(\cos\xi_i)\mathcal{S}(t). \tag{8.3.16}$$

We combine the estimates (8.3.15) and (8.3.16) to obtain

$$\dot{\rho}_i = K(\cos\xi_i)\mathcal{S}(t)\rho_i.$$

□

Remark 8.3.2. *(i) By the uniqueness of the solution for the system of ODEs, one has that*

$$\theta_0(t) \equiv 0 < \theta_1(t) < \ldots < \theta_{N-1}(t) < \theta_N(t) \equiv \pi.$$

(ii) Without symmetry, the relation (8.3.14) *can be simplified as follows. It follows from identity of trigonometric function that*

$$\begin{aligned}
&\cos(\theta_i - \theta_{k+1}) - \cos(\theta_i - \theta_k)\\
&= 2\Big[\cos\Big(\frac{\theta_{k+1} + \theta_k}{2}\Big)\sin\Big(\frac{\theta_{k+1} - \theta_k}{2}\Big)\Big]\sin\theta_i\\
&- 2\Big[\sin\Big(\frac{\theta_{k+1} + \theta_k}{2}\Big)\sin\Big(\frac{\theta_{k+1} - \theta_k}{2}\Big)\Big]\cos\theta_i
\end{aligned}$$

and analogously

$$\begin{aligned}
&\cos(\theta_i - \theta_k) - \cos(\theta_i - \theta_{k-1})\\
&= 2\Big[\cos\Big(\frac{\theta_k + \theta_{k-1}}{2}\Big)\sin\Big(\frac{\theta_k - \theta_{k-1}}{2}\Big)\Big]\sin\theta_i\\
&- 2\Big[\sin\Big(\frac{\theta_k + \theta_{k-1}}{2}\Big)\sin\Big(\frac{\theta_k - \theta_{k-1}}{2}\Big)\Big]\cos\theta_i.
\end{aligned}$$

Thus, we have

$$\dot{\theta}_i = -K\sum_{k=-N}^{-1}\rho_k\left[\cos(\theta_i - \theta_{k+1}) - \cos(\theta_i - \theta_k)\right]$$

$$
\begin{aligned}
&- K \sum_{k=1}^{N} \rho_k \left[\cos(\theta_i - \theta_k) - \cos(\theta_i - \theta_{k-1})\right] \\
&=: -K\mathcal{S}_1 \sin\theta_i + K\mathcal{S}_2 \cos\theta_i
\end{aligned}
$$

with

$$
\begin{aligned}
\mathcal{S}_1 = 2 \sum_{k=-N}^{-1} \rho_k &\left[\cos\left(\frac{\theta_{k+1}+\theta_k}{2}\right) \sin\left(\frac{\theta_{k+1}-\theta_k}{2}\right)\right] \\
&+ 2\sum_{k=1}^{N} \rho_k \left[\cos\left(\frac{\theta_k+\theta_{k-1}}{2}\right) \sin\left(\frac{\theta_k-\theta_{k-1}}{2}\right)\right] \\
\mathcal{S}_2 = 2 \sum_{k=-N}^{-1} \rho_k &\Big[\sin\Big(\frac{\theta_{k+1}+\theta_k}{2}\Big) \sin\Big(\frac{\theta_{k+1}-\theta_k}{2}\Big)\Big] \\
&+ 2\sum_{k=1}^{N} \rho_k \Big[\sin\Big(\frac{\theta_k+\theta_{k-1}}{2}\Big) \sin\Big(\frac{\theta_k-\theta_{k-1}}{2}\Big)\Big].
\end{aligned}
$$

Next, we study the emergence of monotonicity of ρ_i.

Proposition 8.3.2. (Emergence of strict monotonicity of ρ_i) *Let* $(\rho_i(t), \theta_i(t))$ *be a approximate solution with initial data satisfying* (8.3.3)*. Then* ρ_i *is monotonically decreasing in indices:*

$$
\rho_1(t) > \cdots > \rho_N(t), \quad t > 0.
$$

Proof. Note that the initial data satisfy

$$
\begin{aligned}
&\rho_1(0) = \cdots = \rho_{i_0}(0) = \bar{\rho}_1 \quad \text{and} \quad \rho_{i_0+1}(0) = \cdots = \rho_N(0) = \bar{\rho}_2, \\
&\dot{\rho}_i(0+) = K(\cos\xi_i(0))\mathcal{S}(0)\rho_i(0).
\end{aligned} \tag{8.3.17}
$$

Then, we use (8.3.17) and the monotonicity of $\cos(\cdot)$ in $(0, \pi)$:

$$
\cos\xi_i > \cos\xi_{i+1} \quad \text{for } i = 1, \cdots, N,
$$

we have

$$
\dot{\rho}_1(0+) > \cdots > \dot{\rho}_N(0+).
$$

This yields that there exists a positive constant $\delta \ll 1$ such that

$$\rho_1(t) > \rho_2(t) > \cdots > \rho_N(t), \quad t \in (0, \delta). \tag{8.3.18}$$

We now define a set $\mathcal{T}_1$ and its supremum T_1^*:

$$\mathcal{T}_1 := \{T \in (0, \infty) \ : \ \rho_1(t) > \cdots > \rho_N(t), \quad t \in (0, T)\}, \qquad T_1^* := \sup \mathcal{T}_1.$$

Then, it follows from (8.3.18) that the set $\mathcal{T}_1$ is not empty. We claim:

$$T_1^* = \infty. \tag{8.3.19}$$

Proof of claim (8.3.19): Suppose not, i.e., $T_1^* < \infty$ and there exists an index i_* such that

$$\rho_1(t) > \cdots > \rho_N(t), \quad t \in (0, T_1^*), \qquad \rho_{i_*}(T_1^*) = \rho_{i_*+1}(T_1^*). \tag{8.3.20}$$

Therefore it should be

$$\dot{\rho}_{i_*}(T_1^*) \le \dot{\rho}_{i_*+1}(T_1^*). \tag{8.3.21}$$

On the other hand, it follows from Prop. 8.3.1 that we have

$$\begin{aligned}
\dot{\rho}_{i_*}(T_1^*) &= K\rho_{i_*}(T_1^*)(\cos \xi_{i_*}(T_1^*))\mathcal{S}(T_1^*), \\
\dot{\rho}_{i_*+1}(T^*) &= K\rho_{i_*+1}(T_1^*)(\cos \xi_{i+1}(T_1^*))\mathcal{S}(T_1^*).
\end{aligned}$$

We use (8.3.20) and the monotonicity of cosine function to obtain

$$\dot{\rho}_i(T_1^*) > \dot{\rho}_{i+1}(T_1^*).$$

This is contradictory to (8.3.21). Thus, $T_1^* = \infty$ and we have

$$\rho_1(t) > \cdots > \rho_N(t), \quad t \in (0, \infty).$$

$\square$

Remark 8.3.3. *Suppose $(\rho_i(t), \theta_i(t))$ be an approximate solution with symmetric and decreasing initial data:*

$$\theta_i(0) = -\theta_{-i}(0), \rho_i(0) = \rho_{-i}(0) \quad \textit{and} \quad \rho_1(0) \ge \rho_2(0) \ge \cdots \ge \rho_N(0).$$

Then, the argument given in the proof of Proposition 8.3.2 yields the conservation of monotonicity in ρ_i:

$$\rho_1(t) > \rho_2(t) > \cdots > \rho_N(t) \quad \textit{for} \quad t > 0.$$

Lemma 8.3.3. *For $N > 3$, let $\xi_i \in (\theta_{i-1}, \theta_i)$ to be defined by the relation* (8.3.12)*. Then, ξ_i decreases monotonically and $\mathcal{S}$ is strictly positive:*

$$(i)\ \frac{d}{dt}\cos\xi_i(t) > 0, \quad t \geq 0.$$
$$(ii)\ \mathcal{S}(t) = 2\sum_{i=1}^{N}\mathcal{A}_i\cos\xi_i > \mathcal{S}(0) > 0.$$

Proof. *(i)* Recall the defining relation for ξ_i:

$$\cos\xi_i = \frac{\sin\theta_i - \sin\theta_{i-1}}{\theta_i - \theta_{i-1}}, \quad i = 1, \cdots, N.$$

Recalling (8.3.13), and by direct calculation, we have

$$\begin{aligned}
&\frac{d}{dt}\cos\xi_i \\
&= \frac{(\dot\theta_i\cos\theta_i - \dot\theta_{i-1}\cos\theta_{i-1})(\theta_i - \theta_{i-1}) - (\sin\theta_i - \sin\theta_{i-1})(\dot\theta_i - \dot\theta_{i-1})}{(\theta_i - \theta_{i-1})^2} \\
&= \frac{K\mathcal{S}}{(\theta_i - \theta_{i-1})^2} \\
&\quad \times \underbrace{\Big[-(\sin\theta_i\cos\theta_i - \sin\theta_{i-1}\cos\theta_{i-1})(\theta_i - \theta_{i-1}) + (\sin\theta_i - \sin\theta_{i-1})^2 \Big]}_{=:\mathcal{L}}.
\end{aligned}$$

By Proposition 8.3.2, we have

$$\mathcal{S}(t) = 2\sum_{i=1}^{N-1}(\rho_i - \rho_{i+1})\sin\theta_i > 0, \quad t > 0.$$

Thus, it suffices to show the following claim:

$$\mathcal{L}(t) > 0, \quad t > 0, \tag{8.3.22}$$

to derive the desired result.

Proof of claim (8.3.22)*:* For notational simplicity, we set

$$x := \theta_{i-1} \quad \text{and} \quad \varepsilon := \Delta\theta.$$

Then we attain

$$\begin{aligned}\mathcal{L} &= \Big(\sin(x+\varepsilon)-\sin x\Big)^2 - \varepsilon\Big(\sin(x+\varepsilon)\cos(x+\varepsilon)-\sin x\cos x\Big)\\ &= \Big(\sin(x+\varepsilon)-\sin x\Big)^2 - \varepsilon\Big[\big(\sin(x+\varepsilon)-\sin x\big)\big(\cos(x+\varepsilon)+\cos x\big)\\ &\quad - \sin(x+\varepsilon)\cos x + \cos(x+\varepsilon)\sin x\Big]\\ &= \Big(\sin(x+\varepsilon)-\sin x\Big)^2\\ &\quad - \varepsilon\Big[\big(\sin(x+\varepsilon)-\sin x\big)\big(\cos(x+\varepsilon)+\cos x\big) - \sin\varepsilon\Big].\end{aligned} \tag{8.3.23}$$

We will use the following sinusoidal properties:

$$\begin{aligned}\sin(x+\varepsilon)-\sin x &= 2\sin\frac{\varepsilon}{2}\cos\Big(x+\frac{\varepsilon}{2}\Big),\\ \cos(x+\varepsilon)+\cos x &= 2\cos\frac{\varepsilon}{2}\cos(x+\frac{\varepsilon}{2}).\end{aligned} \tag{8.3.24}$$

We apply (8.3.24) into (8.3.23), then we attain

$$\begin{aligned}\mathcal{L} &= 4\sin^2\frac{\varepsilon}{2}\cos^2(x+\frac{\varepsilon}{2}) - \varepsilon\Big[4\cos^2(x+\frac{\varepsilon}{2})\sin\frac{\varepsilon}{2}\cos\frac{\varepsilon}{2} - \sin\varepsilon\Big]\\ &= 4\cos^2(x+\frac{\varepsilon}{2})\sin\frac{\varepsilon}{2}\Big[\sin\frac{\varepsilon}{2} - \varepsilon\cos\frac{\varepsilon}{2}\Big] + \varepsilon\sin\varepsilon.\end{aligned} \tag{8.3.25}$$

Since $N > 3$, we have $\varepsilon = \Delta\theta = \frac{\pi}{N}$ so that

$$\varepsilon > \tan\frac{\varepsilon}{2}, \quad \text{i.e.,} \quad \sin\frac{\varepsilon}{2} - \varepsilon\cos\frac{\varepsilon}{2} < 0. \tag{8.3.26}$$

In (8.3.25), we use (8.3.26) and $\cos^2(x+\frac{\varepsilon}{2}) \le 1$ to find

$$\begin{aligned}\mathcal{L} &\ge 4\sin\frac{\varepsilon}{2}\Big(\sin\frac{\varepsilon}{2} - \varepsilon\cos\frac{\varepsilon}{2}\Big) + \varepsilon\sin\varepsilon\\ &= 4\sin\frac{\varepsilon}{2}\Big(\sin\frac{\varepsilon}{2} - \varepsilon\cos\frac{\varepsilon}{2}\Big) + 2\varepsilon\sin\frac{\varepsilon}{2}\cos\frac{\varepsilon}{2}\\ &= 4\sin\frac{\varepsilon}{2}\Big(\sin\frac{\varepsilon}{2} - \frac{\varepsilon}{2}\cos\frac{\varepsilon}{2}\Big).\end{aligned} \tag{8.3.27}$$

Note that

$$\frac{\varepsilon}{2} < \tan\frac{\varepsilon}{2} \qquad \text{for } 0 < \varepsilon < \frac{\pi}{2}.$$

Thus, we have

$$\sin\frac{\varepsilon}{2} - \frac{\varepsilon}{2}\cos\frac{\varepsilon}{2} > 0.$$

In (8.3.27), $\mathcal{L}$ is positive. Therefore, we have

$$\frac{d}{dt}\cos\xi_i > 0 \quad \text{for } t \geq 0. \tag{8.3.28}$$

(ii) At time $t = 0$, we have

$$\begin{aligned}\mathcal{S}(0) &= 2\sum_{i=1}^{N}\mathcal{A}_i\cos\xi_i(0) = 2\sum_{i=1}^{N}(\rho_i(0) - \rho_{i-1}(0))\sin\theta_i(0)\\ &= 2(\bar{\rho}_1 - \bar{\rho}_2)\sin\theta_{i_0}(0) > 0.\end{aligned}$$

It follows from (8.3.28) that we have

$$\frac{d}{dt}\mathcal{S}(t) = 2\sum_{i=1}^{N}\mathcal{A}_i\frac{d}{dt}\cos\xi_i > 0.$$

This implies

$$\mathcal{S}(t) \geq \mathcal{S}(0) > 0 \quad \text{for} \quad t \geq 0.$$

□

Theorem 8.3.1. *For $N \geq 3$, let (θ_i, ρ_i) be a solution to* (8.3.5) *with initial data* (8.3.3). *Then, there exist positive constants Λ_{1i} and Λ_2 depending on initial data such that*

$$\begin{aligned}&(i) \quad |\theta_i(t)| \leq |\theta_i(0)|e^{-K\Lambda_{1i}t}, \quad i = 1,\cdots,N-1.\\ &(ii) \quad \bar{\rho}_1 e^{\Lambda_2 K t} \leq \|\rho(\cdot,t)\|_\infty \leq \bar{\rho}_1 e^{K(1+2(\bar{\rho}_1-\bar{\rho}_2)\Delta\theta)t}.\end{aligned}$$

Proof. (i) It follows from Proposition 8.3.1 and Lemma 8.3.3 that for $i = 1,\cdots,N-1$,

$$\begin{aligned}\dot{\theta}_i(t) &= -K\mathcal{S}(t)\sin\theta_i(t) \leq -K\mathcal{S}(0)\sin\theta_i(t)\\ &\leq -K\mathcal{S}(0)\frac{\sin\theta_i(0)}{\theta_i(0)}\theta_i(t) =: -K\Lambda_{1i}\theta_i(t),\end{aligned} \tag{8.3.29}$$

where we used the fact that

$$\mathcal{S}(0)\frac{\sin\theta_i(0)}{\theta_i(0)} =: \Lambda_{1i}, \qquad \theta_i(t) \le \theta_i(0) < \pi, \quad \sin\theta_i(t) \ge \frac{\sin\theta_i(0)}{\theta_i(0)}\theta_i(t).$$

Then (8.3.29) implies the exponential decay of θ_i:

$$\theta_i(t) = \theta_i(0)e^{-K\Lambda_{1i}t}, \quad i = 1, \cdots, N-1, \quad t > 0.$$

Since $\xi_i \in (\theta_{i-1}, \theta_i)$, it is clear that ξ_i converges to zero monotonically for $i = 1, \cdots, N-1$ as $t \to \infty$ due to Lemma 8.3.3.

(ii) For the estimate of $||\rho||_{L^\infty(\mathbb{R}_+)}$, it suffices to consider the dynamics of ρ_1 due to Proposition 8.3.2 and even parity of ρ_i. We now consider the dynamics of ρ_1:

$$\dot{\rho}_1 = K\cos\xi_1\mathcal{S}\rho_1. \tag{8.3.30}$$

• (Lower-bound estimate): We use Lemma 8.3.3:

$$\cos\xi_1(t) \ge \cos\xi_1(0), \quad \text{and} \quad \mathcal{S}(t) \ge \mathcal{S}(0) = 2(\bar{\rho}_1 - \bar{\rho}_2)\sin\theta_{i_0}(0) > 0, \quad t \ge 0$$

to find

$$\dot{\rho}_1(t) \ge K\cos\xi_1(0)\mathcal{S}(0)\rho_1(t) =: \Lambda_2\rho_1(t).$$

This yields the desired lower bound estimate for ρ_1.

• (Upper-bound estimate): Using (8.3.4), and for $N \ge 2$, we have

$$\cos\xi_1 \ge \cos\Delta\theta > 0 \quad \text{and} \quad \mathcal{S}(t) = 2\sum_{i=1}^{N}\mathcal{A}_i\cos\xi_i \le 1 + 2\left(\bar{\rho}_1 - \bar{\rho}_2\right)\Delta\theta.$$

In (8.3.30), the above relations imply

$$\dot{\rho}_1 \le K\left(1 + 2\left(\bar{\rho}_1 - \bar{\rho}_2\right)\Delta\theta\right)\rho_1.$$

This yields the desired estimate. □

8.4 Numerical simulations

In this section, we perform several numerical simulations to confirm analytical results discussed in previous sections. We used the front-tracking scheme depicted in Section 8.3 and for the construction of approximate solutions. We employ the fourth-order Runge-Kutta method with time step $h = 0.01$ to implement the numerical simulations.

We consider the Cauchy problem to the continuity equation for identical oscillators with $g(\Omega) = \delta(\Omega)$:

$$\begin{cases} \partial_t \rho - K\partial_\theta(L[\rho]\rho) = 0, \quad (\theta, t) \in \mathbb{T} \times \mathbb{R}_+, \\ \rho(\theta, 0) = \rho_0(\theta). \end{cases}$$

We set the coupling strength $K = 1$ and the common natural frequency $\Omega = 0$. We implement the numerical simulations on the following approximated system given in (8.2.3):

$$\begin{cases} \dot{\theta}_i = -L[\rho](\theta_i), \quad i = 1, \dots, N, \\ \dot{\rho}_i = \rho_i \dfrac{L[\rho](\theta_{i+1}) - L[\rho](\theta_i)}{\theta_{i+1} - \theta_i} \end{cases} \tag{8.4.1}$$

8.4.1 Symmetric and monotone initial data

In this part, we consider an even, periodic and monotone initial datum ρ_0:

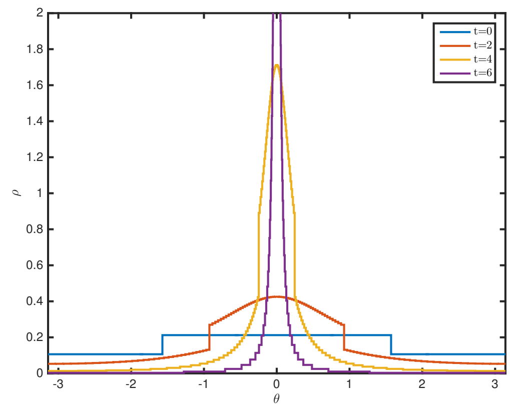
$$\begin{aligned} &\rho_0(\theta + 2\pi) = \rho_0(\theta), \quad \rho_0(-\theta) = \rho_0(\theta), \quad \theta \in (-\pi, \pi) \\ &\rho_0(\theta_1) \geq \rho_0(\theta_2) \quad \text{for} \quad |\theta_1| < |\theta_2|, \quad \theta_1, \theta_2 \in (-\pi, \pi). \end{aligned}$$

To confirm with analytical results in previous section, we consider a non-vacuum two step initial datum:

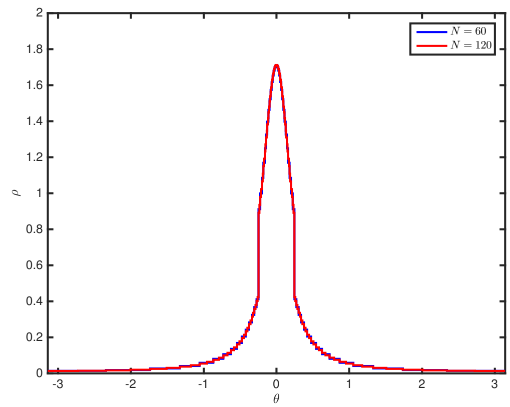
$$\rho_0(\theta) = \begin{cases} \bar{\rho}_1 = \frac{2}{3\pi} & -\frac{\pi}{2} \leq \theta < \frac{\pi}{2}, \\ \bar{\rho}_2 = \frac{1}{3\pi} & \theta \in (-\pi, -\frac{\pi}{2}) \cup [\frac{\pi}{2}, \pi), \end{cases} \tag{8.4.2}$$

With the initial datum (8.4.2), we have

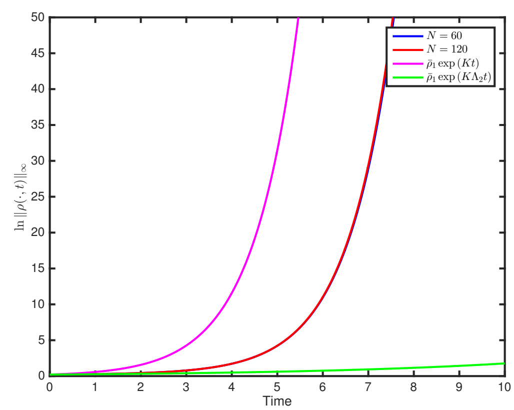
$$\mathcal{S}(0) = 2\int_0^\pi \rho_0(\theta)\cos\theta d\theta = \frac{2}{3\pi} \quad \text{and} \quad \cos\xi_1(0) = \frac{\sin\theta_1(0)}{\theta_1(0)} = \frac{N}{2\pi}\sin\frac{2\pi}{N}.$$



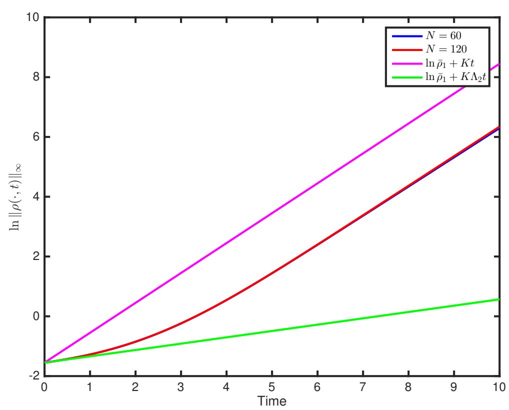
(a) Emergence of synchronization for $N = 120$



(b) Comparison on $N = 60$ and $N = 120$ at time $t = 4$

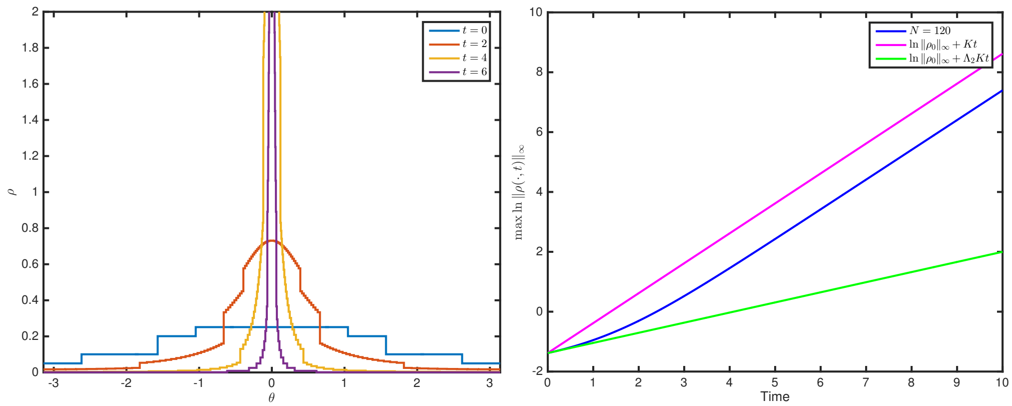


(c) Growth rate of $\|\rho(\cdot, t)\|_\infty$



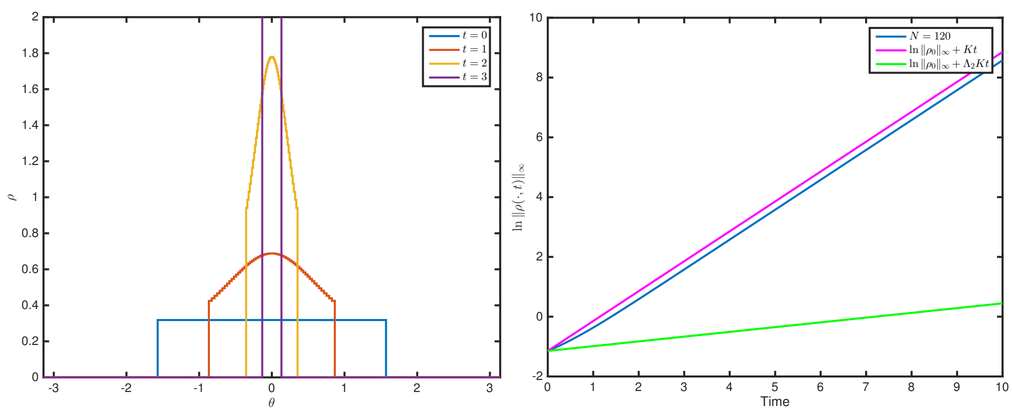
(d) Growth rate of $\ln \|\rho(\cdot, t)\|_\infty$

Figure 8.1: Two step symmetric initial data



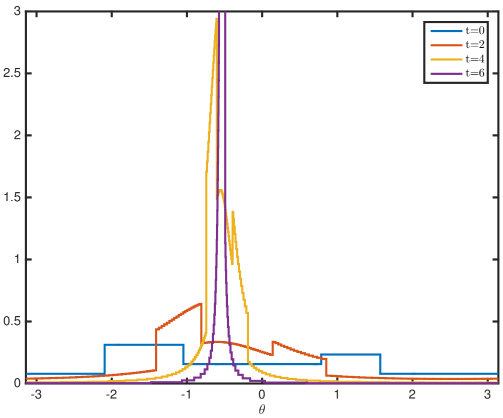
(a) Emergence of synchronization (b) Growth rate of $\ln \|\rho(\cdot, t)\|_\infty$

Figure 8.2: Multi step symmetric initial data

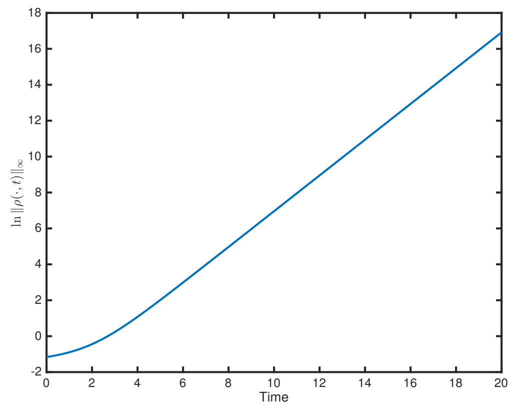


(a) Emergence of synchronization (b) Growth rate of $\ln \|\rho(\cdot, t)\|_\infty$

Figure 8.3: Symmetric initial data with vacuum



(a) Emergence of synchronization



(b) Growth rate of $\ln \|\rho(\cdot,t)\|_\infty$

Figure 8.4: Nonsymmetric initial data

Thus,

$$\Lambda_2 = \frac{N}{3\pi^2}\sin\frac{2\pi}{N} \leq \frac{2}{3\pi} \approx 0.2122.$$

We implement two cases for the number of subintervals to be $N = 60$ and $N = 120$. For $N = 60$ and 120, Λ_2 is determined by $\Lambda_2 \approx 0.2118$ and 0.2121, respectively. In Figure 8.1, we implement the simulation of the dynamics (8.4.1) with initial values as in (8.4.2). In Figure 8.1(a), it shows the emergence of synchronization with symmetric two step initial configuration. We can easily observe that once the monotonicity appears, it preserves as explained in Remark 8.3.3. We compare the growth rate of maximum value of the density. The numerical results in Figure 8.1(c) and 8.1(d) show that the growth rate is uniformly bounded with respect to the number of subintervals for the approximation solutions, which support the estimate of growth rate in Theorem 8.3.1:

$$\bar{\rho}_1 e^{\Lambda_2 K t} \leq \|\rho(\cdot,t)\|_\infty \leq \bar{\rho}_1 e^{Kt}.$$

where the upper bound is determined by the initial datum (8.4.2) and, for the lower bound, we plot with the $\Lambda_2 \approx 0.2118$ which corresponds for the case $N = 60$.

As a second example, we consider a non-vacuum multi-step symmetric and monotone initial datum:

$$\rho_0(\theta) = \begin{cases} 5 \times \frac{3}{19\pi} & [-\frac{\pi}{3}, \frac{\pi}{3}), \\ 4 \times \frac{3}{19\pi} & [-\frac{\pi}{2}, -\frac{\pi}{3}) \cup [\frac{\pi}{3}, \frac{\pi}{2}), \\ 2 \times \frac{3}{19\pi} & [-\frac{5\pi}{6}, -\frac{\pi}{2}) \cup [\frac{\pi}{2}, \frac{5\pi}{6}), \\ 1 \times \frac{3}{19\pi} & (-\pi, \frac{5\pi}{6}) \cup [\frac{5\pi}{6}, \pi). \end{cases}$$

With this initial datum, we have

$$\mathcal{S}(0) = 2\int_0^\pi \rho_0(\theta)\cos\theta \, d\theta = \frac{15 + 3\sqrt{3}}{19\pi} \quad \text{and} \quad \cos\xi_1(0) = \frac{N}{2\pi}\sin\frac{2\pi}{N}.$$

We implement the simulation with $N = 120$, we attain

$$\Lambda_2 = \frac{N(15 + 3\sqrt{3})}{38\pi^2}\sin\frac{2\pi}{N} \approx 0.3382.$$

Numerical results is shown in Figure 8.2. We can see in Figure 8.2(a) that the monotonicity of $\rho(\theta, t)$ in $\theta \in (0, \pi)$ is preserved as we mentioned in Remark 8.3.3. The maximum value $\|\rho(\cdot, t)\|_\infty$ shows exponential growth as in Figure 8.2(b).

As a third example, we also consider the following symmetric datum with vacuum:

$$\rho_0(\theta) = \begin{cases} \frac{1}{4\pi} & -\frac{\pi}{2} \le \theta < \frac{\pi}{2}, \\ 0 & \theta \in (-\pi, -\frac{\pi}{2}) \cup [\frac{\pi}{2}, \pi). \end{cases}$$

In this case, $\mathcal{S}(0)$ becomes $\mathcal{S}(0) = \frac{1}{2\pi}$. Thus, with $N = 120$, we have

$$\Lambda_2 \approx 0.1591.$$

In Figure 8.3, we can see that the density function is initially confined in $(-\frac{\pi}{2}, \frac{\pi}{2})$ and concentrate to the origin exponentially as time increases.

8.4.2 Non-symmetric initial data

In this part, we consider the non-symmetric initial datum. Although we didn't treat this initial setting analytically in previous sections, we implement numerical simulation to compare the evolution of approximated solution. We choose an initial configuration to satisfy

$$\rho_0(\theta) = \begin{cases} 1 \times \frac{12}{49\pi} & (-\pi, -\frac{2\pi}{3}) \cup [\frac{\pi}{2}, \pi), \\ 4 \times \frac{12}{49\pi} & [-\frac{2\pi}{3}, -\frac{\pi}{3}), \\ 2 \times \frac{12}{49\pi} & [-\frac{\pi}{3}, \frac{\pi}{4}), \\ 3 \times \frac{12}{49\pi} & [\frac{\pi}{4}, \frac{\pi}{2}). \end{cases}$$

The numerical implement with $N = 120$ is shown in Figure 8.4(a). As time increases, the density function $\rho(\theta, t)$ makes a spike and the maximum value grows exponentially. On the other hand, the location of phase concentration is not zero.

Chapter 9

Synchronization of the Kuramoto-Sakaguchi equation

In this chapter, we consider the emergence of complete synchronization of kinetic Kuramoto model. We present sufficient conditions which condense the distribution of oscillator by focusing the dynamics of kinetic Kuramoto order parameters. We show the appearance of synchronization for the identical oscillators and the positive invariant behavior for the non-identical case. This chapter is based on the joint work in [35].

9.1 Synchronization of the identical Kuramoto-Sakaguchi equation

In this section, we present an emergent dynamics of the kinetic Kuramoto model with the same natural frequencies. We first introduce order parameters measuring the degree of synchronization for the kinetic model for the following two sections. Following [2, 36], we define real order parameters r and ϕ:

$$re^{\mathrm{i}\phi} := \int_{\mathbb{T}\times\mathbb{R}} e^{\mathrm{i}\theta} f d\Omega d\theta = \int_{\mathbb{T}} e^{\mathrm{i}\theta} \rho_f d\theta. \tag{9.1.1}$$

We devide (9.1.1) by $e^{i\phi}$ on both sides to get

$$r = \int_{\mathbb{T}} e^{i(\theta-\phi)} \rho_f d\theta. \tag{9.1.2}$$

By comparing real and imaginary part on both sides of (9.1.2), we attain

$$r = \int_{\mathbb{T}} \cos(\theta-\phi)\rho_f(\theta,t)d\theta, \qquad 0 = \int_{\mathbb{T}} \sin(\theta-\phi)\rho_f(\theta,t)d\theta. \tag{9.1.3}$$

We use (9.1.3) to see

$$\begin{aligned}
L[f] &= \int_{\mathbb{T}} \sin(\theta-\theta_*)\rho_f(\theta_*,t)d\theta_* \\
&= \int_{\mathbb{T}} \sin\big((\theta-\phi)-(\theta_*-\phi)\big)\rho_f(\theta_*,t)d\theta_* \\
&= \int_{\mathbb{T}} \big(\sin(\theta-\phi)\cos(\theta_*-\phi) - \cos(\theta-\phi)\sin(\theta_*-\phi)\big)\rho_f(\theta_*,t)d\theta_* \\
&= \sin(\theta-\phi)\int_{\mathbb{T}} \cos(\theta_*-\phi)\rho_f(\theta_*,t)d\theta_* \\
&- \cos(\theta-\phi)\int_{\mathbb{T}} \sin(\theta_*-\phi)\rho_f(\theta_*,t)d\theta_* \\
&= r\sin(\theta-\phi).
\end{aligned} \tag{9.1.4}$$

Thus, we combine (2.3.4) and (9.1.4) to rewrite the kinetic K-S equation in terms of order parameters:

$$\partial_t f + \partial_\theta\Big(\Omega f - Kr\sin(\theta-\phi)f\Big) = 0. \tag{9.1.5}$$

Without loss generality, we can assume that the $g(\Omega) = \delta(\Omega)$ and we set

$$f(\theta,\Omega,t) = \rho_f(\theta,t)\delta(\Omega).$$

As long as there is no confusion, we suppress f-dependence in ρ_f, i.e., we set

$$\rho := \rho_f.$$

Then, the local mass density ρ_f satisfies

$$\begin{aligned} &\partial_t \rho + \partial_\theta [v(\rho)\rho] = 0, \quad \theta \in \mathbb{T},\ t > 0, \\ &v(\rho) = -K \int_{\mathbb{T}} \sin(\theta - \theta_*)\rho(\theta_*, t) d\theta_*. \end{aligned} \tag{9.1.6}$$

or equivalently,

$$\partial_t \rho - \partial_\theta \Big(Kr \sin(\theta - \phi)\rho \Big) = 0. \tag{9.1.7}$$

In the following two subsections, we will present the order parameter approach to show the formation of completely synchronized states from non-completely synchronized states.

9.1.1 Dynamics of order parameters

In this subsection, we study the dynamics of the order parameters introduced in previous section. We now consider the dynamics of order parameters. We differentiate the relation (9.1.1) with respect to t to obtain

$$\dot{r} e^{\mathrm{i}\phi} + \mathrm{i} r \dot{\phi} e^{\mathrm{i}\phi} = \int_{\mathbb{T}} \partial_t \rho e^{\mathrm{i}\theta} d\theta. \tag{9.1.8}$$

We devide (9.1.8) by $e^{i\phi}$ on both sides to get

$$\dot{r} + \mathrm{i} r \dot{\phi} = \int_{\mathbb{T}} \partial_t \rho e^{\mathrm{i}(\theta - \phi)} d\theta. \tag{9.1.9}$$

We again compare real and imaginary parts of (9.1.9) and employ (9.1.5) to derive a coupled dynamical system for r and ϕ in the following lemma.

Proposition 9.1.1. *Let ρ be a solution to* (9.1.6) *and r, ϕ are order parameters defined by the relation* (9.1.2). *Then, r and ϕ satisfy*

$$\begin{aligned} &(i)\ \dot{r} = Kr \int_{\mathbb{T}} \sin^2(\theta - \phi)\rho(\theta, t) d\theta, \\ &(ii)\ \dot{\phi} = -K \int_{\mathbb{T}} \sin(\theta - \phi)\cos(\theta - \phi)\rho(\theta, t) d\theta, \\ &(iii)\ \ddot{r} = \frac{(\dot{r})^2}{r} + 2r(\dot{\phi})^2 - 2(Kr)^2 \int_{\mathbb{T}} \sin^2(\theta - \phi)\cos(\theta - \phi)\rho(\theta, t) d\theta. \end{aligned}$$

,

Proof. Recall that r and ϕ satisfy

$$r = \int_{\mathbb{T}} \cos(\theta - \phi)\rho(\theta, t)d\theta, \qquad 0 = \int_{\mathbb{T}} \sin(\theta - \phi)\rho(\theta, t)d\theta.$$

(i) We differentiate the above relation for r with respect to t and obtain

$$\begin{aligned} \dot{r} &= \int_{\mathbb{T}} \cos(\theta - \phi)\partial_t \rho d\theta = Kr \int_{\mathbb{T}} \cos(\theta - \phi)\partial_\theta \Big[\rho(\theta, t)\sin(\theta - \phi)\Big] d\theta \\ &= Kr \int_{\mathbb{T}} \sin^2(\theta - \phi)\rho(\theta, t)d\theta. \end{aligned} \tag{9.1.10}$$

(ii) Again, it follows from (9.1.7) that we have

$$\begin{aligned} \dot{\phi} &= \frac{1}{r}\int_{\mathbb{T}} \sin(\theta - \phi)\partial_t \rho(\theta, t)d\theta \\ &= K \int_{\mathbb{T}} \sin(\theta - \phi)\partial_\theta \Big[\rho(\theta, t)\sin(\theta - \phi)\Big] d\theta \\ &= -K \int_{\mathbb{T}} \sin(\theta - \phi)\cos(\theta - \phi)\rho(\theta, t)d\theta, \end{aligned} \tag{9.1.11}$$

where we take integration by parts for the last equalities in (9.1.10) and (9.1.11).

(iii) We now take time derivative on (9.1.10) to get the second order equation:

$$\begin{aligned} \ddot{r} &= K\dot{r} \int_{\mathbb{T}} \sin^2(\theta - \phi)\rho(\theta, t)d\theta - 2Kr\dot{\phi} \int_{\mathbb{T}} \sin(\theta - \phi)\cos(\theta - \phi)\rho(\theta, t)d\theta \\ &\quad + Kr \int_{\mathbb{T}} \sin^2(\theta - \phi)\partial_t \rho(\theta, t)d\theta \\ &= \frac{(\dot{r})^2}{r} + 2r(\dot{\phi})^2 + (Kr)^2 \int_{\mathbb{T}} \sin^2(\theta - \phi)\partial_\theta \Big[\rho(\theta, t)\sin(\theta - \phi)\Big] d\theta \\ &= \frac{(\dot{r})^2}{r} + 2r(\dot{\phi})^2 - (Kr)^2 \int_{\mathbb{T}} 2\sin^2(\theta - \phi)\cos(\theta - \phi)\rho(\theta, t)d\theta. \end{aligned}$$

□

The following lemma is an analogy of Lemma 3.2 of [36]

Lemma 9.1.1. *Let ρ be a solution to* (9.1.6) *with $r_0 > 0$ and $\rho_0 \in L^1(\mathbb{T})$. Then, there exists a positive constant $r^\infty \le 1$ such that*

$$(i)\ \inf_{0\le t<\infty} r(t) \ge r_0 > 0, \qquad \lim_{t\to\infty}(r(t),\dot{r}(t)) = (r^\infty, 0),$$
$$(ii)\ |\dot{\phi}| \le K(1-r), \quad \lim_{t\to\infty}\dot{\phi}(t) = 0.$$

Proof. (i) It follows from Proposition 9.1.1 that we have

$$\dot{r} = Kr\int_{\mathbb{T}} \sin^2(\theta-\phi)\rho(\theta,t)d\theta \ge 0, \quad \text{i.e.,} \quad r(t) \ge r_0. \tag{9.1.12}$$

On the other hand, due to the uniform boundedness of r, we can see that the order parameter r converges to $r^\infty \le 1$ and $\dot{r}$ converges to zero as time $t \to \infty$.

(ii) We need to show that

$$-K(1-r) \le \dot{\phi} \le K(1-r).$$

• Case A (Upper bound): We use the second result in Proposition 9.1.1 to obtain

$$\begin{aligned}
\dot{\phi} &= -K\int_{\mathbb{T}} \sin(\theta-\phi)\cos(\theta-\phi)\rho(\theta,t)d\theta \\
&= -K\int_{\mathbb{T}} \underbrace{\big(\sin(\theta-\phi)-1\big)\big(\cos(\theta-\phi)-1\big)}_{\ge 0}\rho(\theta,t) \\
&+ \big(\cos(\theta-\phi)+\sin(\theta-\phi)-1\big)\rho(\theta,t)d\theta \\
&\le -K\int_{\mathbb{T}} \cos(\theta-\phi)\rho(\theta,t)d\theta - K\int_{\mathbb{T}} \sin(\theta-\phi)\rho(\theta,t)d\theta + K \\
&= -Kr + K = K(1-r).
\end{aligned}$$

• Case B (Lower bound): Similar to Case A, we have

$$\begin{aligned}
\dot{\phi} &= -K\int_{\mathbb{T}} \sin(\theta-\phi)\cos(\theta-\phi)\rho(\theta,t)d\theta \\
&= -K\int_{\mathbb{T}} \underbrace{\big(\sin(\theta+\phi)+1\big)\big(\cos(\theta-\phi)-1\big)}_{\le 0}\rho(\theta,t)
\end{aligned}$$

$$
\begin{aligned}
& - \quad \big(\cos(\theta-\phi) - \sin(\theta-\phi) - 1\big)\rho(\theta,t)d\theta \\
& \geq \quad K\int_{\mathbb{T}} \cos(\theta-\phi)\rho(\theta)d\theta - K\int_{\mathbb{T}} \sin(\theta-\phi)\rho(\theta)d\theta - K \\
& = \quad Kr - K = -K(1-r).
\end{aligned}
$$

Finally, we combine Case A and Case B to obtain the desired result.

For the second part, we use Cauchy-Schwarz' inequality, (9.1.13) and the representation formula for $\dot{\phi}$ in Proposition 9.1.1 to obtain

$$
\begin{aligned}
|\ \dot{\phi}\ | &\leq K\int_{\mathbb{T}} |\sin(\theta-\phi)|\ \rho d\theta \leq K\bigg(\int_{\mathbb{T}} |\sin(\theta-\phi)|^2\ \rho d\theta\bigg)^{\frac{1}{2}} ||\rho_0||_{L^1}^{\frac{1}{2}} \\
&\leq \frac{\sqrt{K||\rho_0||_{L^1}}}{\sqrt{r_0}}\sqrt{|\dot{r}|}.
\end{aligned}
$$

We now use the above relation and the result (i) on zero convergence of $\dot{r}$ to conclude

$$
\lim_{t\to\infty} |\dot{\phi}(t)| = 0.
$$

□

9.1.2 Emergence of the complete synchronization

For a given $\delta > 0$, we define a set I_δ:

$$
\begin{aligned}
& I_\delta := I_\delta^+ \cup I_\delta^-, \\
& I_\delta^+ := \{\theta \in \mathbb{T} : |\theta - \phi| < \delta\}, \quad I_\delta^- := \{\theta \in \mathbb{T} : |\theta - (\phi+\pi)| < \delta\}.
\end{aligned} \tag{9.1.13}
$$

Lemma 9.1.2. *Let ρ be a solution to (9.1.6) with $r_0 > 0$ and $\rho_0 \in L^1(\mathbb{T})$. Then, for any $\varepsilon > 0$, there exists a finite time $t_* > 0$ such that*

$$
\int_{\mathbb{T}\setminus I_\delta} \rho(\theta,t)d\theta < \varepsilon, \quad t > t_*.
$$

Proof. Let ε and δ be given positive numbers. Since $\dot{r}$ converges to zero, there exists a positive time $t_* = t_*(\varepsilon,\delta)$ such that

$$
\dot{r}(t) = Kr\int_{\mathbb{T}} \sin^2(\theta-\phi)\rho(\theta,t)d\theta < \delta^2\varepsilon, \quad t > t_*
$$

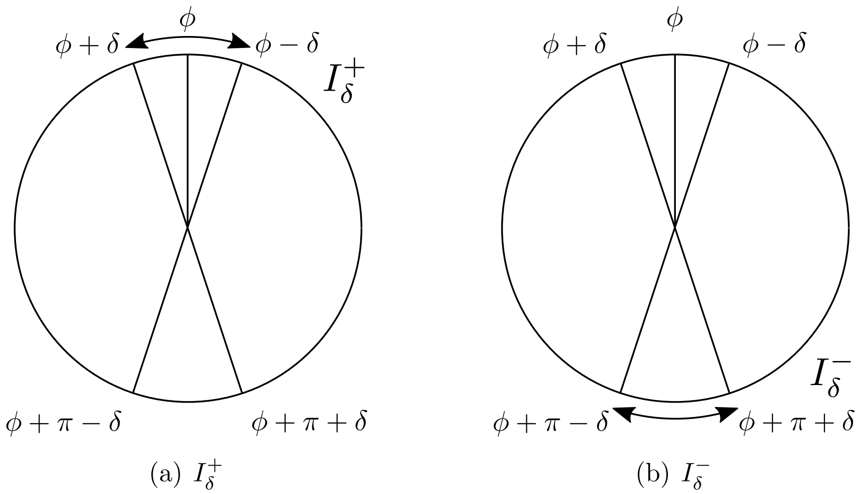


Figure 9.1: Geometric descriptions of I_δ^+ and I_δ^-

Note that for $\theta \in I_\delta$, we have

$$|\sin(\theta-\phi)| < \delta.$$

Then, we have

$$\begin{aligned}\int_{\mathbb{T}\setminus I_\delta} \delta^2 \rho(\theta,t)d\theta &< \int_{\mathbb{T}\setminus I_\delta} \sin^2(\theta-\phi)\rho(\theta,t)d\theta \\ &\le \int_{\mathbb{T}} \sin^2(\theta-\phi)\rho(\theta,t)d\theta \le \delta^2\varepsilon, \quad t \ge t_*.\end{aligned}$$

This implies the desired result. □

Before we present a result on the convergence of f toward the complete synchronized state (mono-cluster state), we briefly discuss its heuristics. Suppose that the initial datum f_0 is absolutely continuous and $r_0 > 0$. Then, since r is bounded and monotone increasing, $\dot{r} \to 0$ as $t \to \infty$. (see Lemma 9.1.1). On the other hand, it follows from Lemma 9.1.2 and estimates on the mean-field limit that the limiting behavior of f will be either a complete synchronized state δ_{ϕ^∞}, $\phi^\infty = \lim_{t\to\infty}\phi(t)$ or bi-polar state $\frac{1}{2}\delta_{\phi^\infty} + \frac{1}{2}\delta_{\phi^\infty+\pi}$. In the following proposition, we show that the bipolar state is not possible asymptotically. For this, we show that the mass near the point $\phi+\pi$ is exponentially decaying to zero.

Proposition 9.1.2. ***(Convergence)*** *Suppose that the coupling strength K and absolutely continuous ρ_0 satisfy*

$$K > 0, \qquad r_0 > 0,$$

and let $\rho = \rho(\theta, t)$ be a solution to (9.1.6) *with initial datum ρ_0. Then, there exist positive numbers T_0 and δ such that*

$$\int_{\phi(t)+\pi-\delta}^{\phi(t)+\pi+\delta} |\rho(\theta,t)|^2 d\theta \le e^{-r\cos\delta K(t-T_1)} \int_{\phi(T_1)+\pi-\delta}^{\phi(T_1)+\pi+\delta} |\rho(\theta,T_1)|^2 d\theta, \qquad \forall\ t > T_1.$$

Proof. Note that for $\delta \in (0,1)$ and t,

$$\theta \in (\phi+\pi-\delta, \phi+\pi+\delta) \quad \Longrightarrow \quad \cos(\theta-\phi) < -\cos\delta. \tag{9.1.14}$$

On the other hand, since $\lim_{t\to\infty} \dot{\phi} = 0$, for any $\varepsilon \in (0, K)$, there exist $T_1 = T_1(\varepsilon, \delta) > 0$ such that

$$|\ \dot{\phi}\ | < \varepsilon r_0 \sin\delta, \qquad \forall\ t > T_1. \tag{9.1.15}$$

We next claim: the functional $\Lambda(t) := \int_{\phi(t)+\pi-\delta}^{\phi(t)+\pi+\delta} |\rho(\theta,t)|^2 d\theta$ satisfies a Gronwall's inequality:

$$\frac{d\Lambda(t)}{dt} \le -\lambda r_0 \Lambda(t), \quad t \ge T_1. \tag{9.1.16}$$

Proof of claim (9.1.16): By direct calculation, we have

$$\begin{aligned}
\frac{d\Lambda}{dt} &= \dot{\phi}\rho^2(\phi+\pi+\delta) - \dot{\phi}\rho^2(\phi+\pi-\delta) + 2\int_{\phi+\pi-\delta}^{\phi+\pi+\delta} \rho\partial_t\rho d\theta \\
&= \dot{\phi}\rho^2(\phi+\pi+\delta) - \dot{\phi}\rho^2(\phi+\pi-\delta) + 2rK\int_{\phi+\pi-\delta}^{\phi+\pi+\delta} \rho\partial_\theta\Big[\sin(\theta-\phi)\rho\Big] d\theta \\
&= \dot{\phi}\rho^2(\phi+\pi+\delta) - \dot{\phi}\rho^2(\phi+\pi-\delta) - 2rK\sin\delta\rho^2(\phi+\pi+\delta) \\
&- 2rK\sin\delta\rho^2(\phi+\pi-\delta) - rK\int_{\phi+\pi-\delta}^{\phi+\pi+\delta} \sin(\theta-\phi)\partial_\theta(\rho^2) d\theta \\
&= \dot{\phi}\rho^2(\phi+\pi+\delta) - \dot{\phi}\rho^2(\phi+\pi-\delta) - rK\sin\delta\rho^2(\phi+\pi+\delta) \\
&- rK\sin\delta\rho^2(\phi+\pi-\delta) + rK\int_{\phi+\pi-\delta}^{\phi+\pi+\delta} \cos(\theta-\phi)\rho^2 d\theta
\end{aligned}$$

$$
\begin{aligned}
&= \Big(\dot{\phi} - rK\sin\delta\Big)\rho^2(\phi+\pi+\delta) - \Big(\dot{\phi} + rK\sin\delta\Big)\rho^2(\phi+\pi-\delta) \\
&+ rK\int_{\phi+\pi-\delta}^{\phi+\pi+\delta}\cos(\theta-\phi)\rho^2 d\theta. \qquad (9.1.17)
\end{aligned}
$$

On the other hand, it follows from (9.1.14) and (9.1.15) to see that

$$
\begin{aligned}
&\dot{\phi} - rK\sin\delta \le (\varepsilon r_0 - rK)\sin\delta \le (\varepsilon - K)r_0\sin\delta < 0, \\
&\dot{\phi} + rK\sin\delta > (-\varepsilon r_0 + rK)\sin\delta > (K-\varepsilon)r_0\sin\delta > 0, \\
&\int_{\phi+\pi-\delta}^{\phi+\pi+\delta}\cos(\theta-\phi)\rho^2 d\theta \le -\cos\delta\Lambda(t).
\end{aligned} \qquad (9.1.18)
$$

Finally, we combine (9.1.17) and (9.1.18) to obtain a Gronwall's inequality:

$$
\frac{d\Lambda(t)}{dt} \le -r\cos\delta K\Lambda(t), \quad t \ge T_1.
$$

This yields the desired estimate. □

9.2 Positive invariant property of non-identical oscillators

In this section, we define the order parameters for non-identical oscillators. For given $\bar{\Omega}$, let $f_{\bar{\Omega}}(\theta,t)$ be the conditional distribution corresponding to $\bar{\Omega}$ so that

$$
f(\theta,\bar{\Omega},t) = g(\bar{\Omega})f_{\bar{\Omega}}(\theta,t) \quad \text{and} \quad \int_{\mathbb{T}} f_{\bar{\Omega}}(\theta,t)d\theta = 1. \qquad (9.2.19)
$$

Define local order parameters $r_{\bar{\Omega}}$ and $\phi_{\bar{\Omega}}$ for $\bar{\Omega}$ to satisfy

$$
r_{\bar{\Omega}}e^{i\phi_{\bar{\Omega}}} := \int_{\mathbb{T}} f_{\bar{\Omega}}(\theta,t)e^{i\theta}d\theta \qquad (9.2.20)
$$

By the same argument in previous section, we have

$$
r_{\bar{\Omega}} = \int_{\mathbb{T}} f_{\bar{\Omega}}(\theta,t)\cos(\theta-\phi_{\bar{\Omega}})d\theta, \quad \text{and} \quad 0 = \int_{\mathbb{T}} f_{\bar{\Omega}}(\theta,t)\sin(\theta-\phi_{\bar{\Omega}})d\theta.
$$

We also define the global order parameters r and ϕ to satisfy

$$\begin{aligned} re^{i\phi} &:= \int_{\mathbb{R}}\int_{\mathbb{T}} f(\theta,\Omega,t)e^{i\theta}d\theta d\Omega = \int_{\mathbb{R}}\int_{\mathbb{T}} g(\Omega)f_\Omega(\theta,t)e^{i\theta}d\theta d\Omega \\ &= \int_{\mathbb{R}} g(\Omega)\int_{\mathbb{T}} f_\Omega(\theta,t)e^{i\theta}d\theta d\Omega = \int_{\mathbb{R}} g(\Omega)r_\Omega e^{i\phi_\Omega}d\Omega. \end{aligned} \tag{9.2.21}$$

Then, we have

$$\begin{aligned} r &= \int_{\mathbb{R}}\int_{\mathbb{T}} \cos(\theta-\phi)f(\theta,\Omega,t)d\theta d\Omega \\ &= \int_{\mathbb{R}} g(\Omega)r_\Omega\cos(\phi_\Omega-\phi)d\Omega \end{aligned} \quad \text{and} \quad \begin{aligned} 0 &= \int_{\mathbb{R}}\int_{\mathbb{T}} \sin(\theta-\phi)f(\theta,\Omega,t)d\theta d\Omega \\ &= \int_{\mathbb{R}} g(\Omega)r_\Omega\sin(\phi_\Omega-\phi)d\Omega \end{aligned} \tag{9.2.22}$$

By applying (9.2.19) to the kinetic Kuramoto system (2.3.4), we can deduce the equation for the conditional distribution $f_\Omega = f_\Omega(\theta,t)$:

$$\partial_t f_\Omega + \Omega\partial_\theta f_\Omega(\theta,t) - K\partial_\theta\Big[f_\Omega(\theta,t)\int_{\mathbb{R}}\int_{\mathbb{T}}\sin(\theta-\theta_*)g(\Omega_*)f_{\Omega_*}(\theta_*,t)d\theta_* d\Omega_*\Big] = 0 \tag{9.2.23}$$

By direct calculation, we have

$$\begin{aligned} &\int_{\mathbb{R}}\int_{\mathbb{T}}\sin(\theta-\theta_*)g(\Omega_*)f_{\Omega_*}(\theta_*,t)d\theta_* d\Omega_* \\ &= \int_{\mathbb{R}}\int_{\mathbb{T}}\sin\big((\theta-\phi)-(\theta_*-\phi)\big)f(\theta_*,\Omega_*,t)d\theta_* d\Omega_* \\ &= \int_{\mathbb{R}}\int_{\mathbb{T}}\big(\sin(\theta-\phi)\cos(\theta_*-\phi)-\cos(\theta-\phi)\sin(\theta_*-\phi)\big)f(\theta_*,\Omega_*,t)d\theta_* d\Omega_* \\ &= \sin(\theta-\phi)\underbrace{\int_{\mathbb{R}}\int_{\mathbb{T}}\cos(\theta_*-\phi)f(\theta_*,\Omega_*,t)d\theta_* d\Omega_*}_{=r} \\ &\qquad - \cos(\theta-\phi)\underbrace{\int_{\mathbb{R}}\int_{\mathbb{T}}\sin(\theta_*-\phi)f(\theta_*,\Omega_*,t)d\theta_* d\Omega_*}_{=0} \\ &= r\sin(\theta-\phi), \end{aligned}$$

where we use (9.2.22) in the last equality. So, the equation (9.2.23) can be rewritten as

$$\partial_t f_\Omega + \Omega\partial_\theta f_\Omega(\theta,t) - Kr\partial_\theta\Big[f_\Omega(\theta,t)\sin(\theta-\phi)\Big] = 0,$$

or equivalently,

$$\partial_t f_\Omega + \partial_\theta\Big[f_\Omega(\theta,t)\big(\Omega - Kr\sin(\theta-\phi)\big)\Big] = 0. \tag{9.2.24}$$

We also rewrite (2.3.4) with the global order parameters:

$$\partial_t f(\theta,\Omega,t) + \partial_\theta\Big[f(\theta,\Omega,t)\big(\Omega - Kr\sin(\theta-\phi)\big)\Big] = 0$$

By taking time derivative on (9.2.20), we attain the dynamics of local order parameters:

$$\begin{aligned}
\dot r_\Omega &= \int_{\mathbb{T}} \cos(\theta-\phi_\Omega)\partial_t f_\Omega(\theta,t)d\theta \\
&= -\int_{\mathbb{T}} \cos(\theta-\phi_\Omega)\partial_\theta\Big[f_\Omega(\theta,t)\big(\Omega - Kr\sin(\theta-\phi)\big)\Big]d\theta \\
&= -\int_{\mathbb{T}} \sin(\theta-\phi_\Omega)f_\Omega(\theta,t)\big(\Omega - Kr\sin(\theta-\phi)\big)d\theta, \\
\dot\phi_\Omega &= \frac{1}{r_\Omega}\int_{\mathbb{T}} \sin(\theta-\phi_\Omega)\partial_t f_\Omega(\theta,t)d\theta \\
&= -\frac{1}{r_\Omega}\int_{\mathbb{T}} \sin(\theta-\phi_\Omega)\partial_\theta\Big[f_\Omega(\theta,t)\big(\Omega - Kr\sin(\theta-\phi)\big)\Big]d\theta \\
&= \frac{1}{r_\Omega}\int_{\mathbb{T}} \cos(\theta-\phi_\Omega)f_\Omega(\theta,t)\big(\Omega - Kr\sin(\theta-\phi)\big)d\theta
\end{aligned}$$

where we take integration by parts in the last equalities. With same argument on the global order parameter (9.2.21), we attain

$$\begin{aligned}
\dot r &= \int_{\mathbb{R}}\int_{\mathbb{T}} \cos(\theta-\phi)\partial_t f(\theta,\Omega,t)d\theta d\Omega \\
&= -\int_{\mathbb{R}}\int_{\mathbb{T}} \cos(\theta-\phi)\partial_\theta\Big[f(\theta,\Omega,t)\big(\Omega - Kr\sin(\theta-\phi)\big)\Big]d\theta d\Omega \\
&= -\int_{\mathbb{R}}\int_{\mathbb{T}} \sin(\theta-\phi)f(\theta,\Omega,t)\big(\Omega - Kr\sin(\theta-\phi)\big)d\theta d\Omega, \\
\dot\phi &= \frac{1}{r}\int_{\mathbb{R}}\int_{\mathbb{T}} \sin(\theta-\phi)\partial_t f(\theta,\Omega,t)d\theta d\Omega \\
&= -\frac{1}{r}\int_{\mathbb{R}}\int_{\mathbb{T}} \sin(\theta-\phi)\partial_\theta\Big[f(\theta,\Omega,t)\big(\Omega - Kr\sin(\theta-\phi)\big)\Big]d\theta d\Omega
\end{aligned}$$

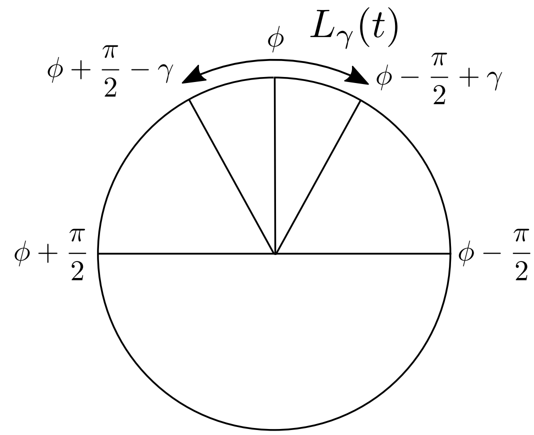


Figure 9.2: Geometric description of $L_\gamma(t)$

$$= \frac{1}{r}\int_{\mathbb{R}}\int_{\mathbb{T}} \cos(\theta-\phi) f(\theta,\Omega,t)\big(\Omega - Kr\sin(\theta-\phi)\big)d\theta d\Omega.$$

For notational simplicity, we set

$$\mathcal{F}_f(S) := \int_{\mathbb{R}}\int_S f(\theta,\Omega,t)d\theta d\Omega$$

for $S \subset \mathbb{T}$. Obviously, we have $\mathcal{F}_f(\mathbb{T}) = 1$. For $0 < \varepsilon < \frac{3\sqrt{3}}{4} - 1$, we choose a positive constant γ such that

$$\frac{\pi}{3} < \gamma < \arcsin\Big(1 - \frac{2\varepsilon}{2\sqrt{3}+1}\Big). \tag{9.2.25}$$

Define a time depending set L_γ to be

$$L_\gamma(t) := \Big(\phi(t) - \frac{\pi}{2} + \gamma, \phi(t) + \frac{\pi}{2} - \gamma\Big).$$

(see Figure 9.2)

For notational simplicity, we denote the boundary values

$$B_{1,\bar\Omega}(t) := f_{\bar\Omega}(\phi(t) - \frac{\pi}{2} + \gamma, t) \quad \text{and} \quad B_{2,\bar\Omega}(t) := f_{\bar\Omega}(\phi(t) + \frac{\pi}{2} - \gamma, t).$$

For every $\bar\Omega \in \operatorname{supp} g(\Omega)$, we assume that

$$\int_{L_\gamma(0)} f_{\bar\Omega}(\theta, 0)d\theta \geq \frac{2+\varepsilon+\cos\gamma}{(1+\sin\gamma)(1+\cos\gamma)}. \tag{9.2.26}$$

If $\int_{L_\gamma(t)} f_{\bar{\Omega}}(\theta,t)d\theta \geq \frac{2+\varepsilon+\cos\gamma}{(1+\sin\gamma)(1+\cos\gamma)}$ for each $\bar{\Omega}$ at time $t \geq 0$, Then,

$$\begin{aligned}
\mathcal{F}_f\big(L_\gamma(t)\big) &= \int_{\mathbb{R}}\int_{L_\gamma(t)} f(\theta,\Omega,t)d\theta, d\Omega \\
&= \int_{\mathbb{R}} g(\Omega)\int_{L_\gamma(t)} f_\Omega(\theta,t)d\theta d\Omega \\
&\geq \int_{\mathbb{R}} g(\Omega)\frac{2+\varepsilon+\cos\gamma}{(1+\sin\gamma)(1+\cos\gamma)}d\Omega \\
&= \frac{2+\varepsilon+\cos\gamma}{(1+\sin\gamma)(1+\cos\gamma)}\int_{\mathbb{R}} g(\Omega)d\Omega = \frac{2+\varepsilon+\cos\gamma}{(1+\sin\gamma)(1+\cos\gamma)}.
\end{aligned}$$

Thus, the assumption (9.2.26) implies

$$\mathcal{F}_f\big(L_\gamma(0)\big) \geq \frac{2+\varepsilon+\cos\gamma}{(1+\cos\gamma)(1+\sin\gamma)} \tag{9.2.27}$$

From (9.2.25) and $0<\varepsilon<\frac{3\sqrt{3}}{4}-1$, we have $1+\varepsilon>\varepsilon(1+\cos\gamma)$ so that

$$\frac{2+\varepsilon+\cos\gamma}{(1+\sin\gamma)(1+\cos\gamma)} > \frac{1+\varepsilon}{1+\sin\gamma} \tag{9.2.28}$$

We will show that

$$\frac{2+\varepsilon+\cos\gamma}{(1+\cos\gamma)(1+\sin\gamma)} \leq 1. \tag{9.2.29}$$

For $\frac{\sqrt{3}}{2}<x<1$, we have the following equlaity:

$$x(1+\sqrt{1-x^2})-1 > -\frac{1+2\sqrt{3}}{2}(x-1).$$

(see Figure 9.3) Thus, we have

$$x(1+\sqrt{1-x^2})-1 > \varepsilon$$

for $\frac{\sqrt{3}}{2}<x<1-\frac{2\varepsilon}{1+2\sqrt{3}}$. By the assumption (9.2.25),

$$\frac{\sqrt{3}}{2} < \sin\gamma < 1-\frac{2\varepsilon}{2\sqrt{3}+1}$$

Hence,

$$\sin\gamma(1+\cos\gamma) > 1+\varepsilon,$$

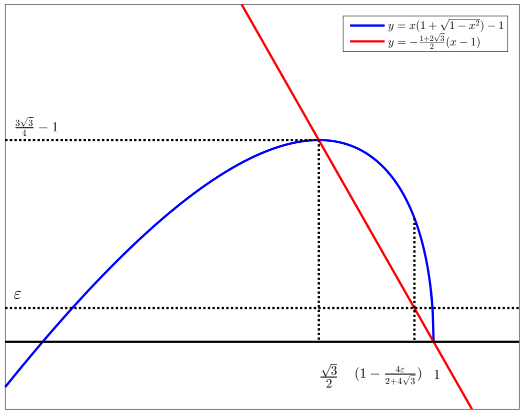


Figure 9.3: $x(1+\sqrt{1-x^2}) - 1 > -\frac{1+2\sqrt{3}}{2}(x-1)$

equivalently,

$$(1+\sin\gamma)(1+\cos\gamma) > 2+\varepsilon+\cos\gamma,$$

which implies (9.2.29).

Lemma 9.2.1. *Under the assumption* (9.2.26)*, we have the following conditions:*

$$|\phi(0) - \phi_{\bar{\Omega}}(0)| < \frac{\pi}{2} - \frac{1+\varepsilon}{1+\cos\gamma}$$

for all $\bar{\Omega} \in suppg(\Omega)$.

Proof. By the definition of the local order parameter (9.2.20), we have

$$\begin{aligned}
&r_{\bar{\Omega}}(0)\cos(\phi_{\bar{\Omega}}(0) - \phi(0)) \\
&= \int_{L_\gamma(0)} \cos(\theta - \phi(0)) f_{\bar{\Omega}}(\theta, 0) d\theta + \int_{\mathbb{T}\setminus L_\gamma(0)} \cos(\theta - \phi(0)) f_{\bar{\Omega}}(\theta, 0) d\theta \\
&\geq \cos(\frac{\pi}{2} - \gamma) \int_{L_\gamma(0)} f_{\bar{\Omega}}(\theta, 0) d\theta - \int_{\mathbb{T}\setminus L_\gamma(0)} f_{\bar{\Omega}}(\theta, 0) d\theta \\
&\geq \sin\gamma \frac{2+\varepsilon+\cos\gamma}{(1+\sin\gamma)(1+\cos\gamma)} - \Big(1 - \frac{2+\varepsilon+\cos\gamma}{(1+\sin\gamma)(1+\cos\gamma)}\Big) \\
&= \frac{2+\varepsilon+\cos\gamma}{1+\cos\gamma} - 1 \\
&= \frac{1+\varepsilon}{1+\cos\gamma} > 0.
\end{aligned}$$

Since $r_{\bar{\Omega}} \le 1$,

$$\cos(\phi_{\bar{\Omega}} - \phi) > \frac{1+\varepsilon}{1+\cos\gamma} > \sin\frac{1+\varepsilon}{1+\cos\gamma} = \cos(\frac{\pi}{2} - \frac{1+\varepsilon}{1+\cos\gamma}).$$

Therefore, $|\phi(0) - \phi_{\bar{\Omega}}(0)| < \frac{\pi}{2} - \frac{1+\varepsilon}{1+\cos\gamma}$. □

Lemma 9.2.2 (Estimates on $\dot{\phi}$ and r)**.** *Suppose that suppg*$(\Omega) \subset (-M, M)$*. We have the following estimates for* $\dot{\phi}$ *and* r*:*

$$\begin{aligned}
&|\dot{\phi}| \le \frac{M}{r} + K(1-r),\\
&r(t) \ge (1+\sin\gamma)\mathcal{F}_f\big(L_\gamma(t)\big) - 1.
\end{aligned}$$

In particular, we have

$$r_0 \ge (1+\sin\gamma)\mathcal{F}_f\big(L_\gamma(0)\big) - 1 > \varepsilon > 0.$$

Proof. • (Estimates for $\dot{\phi}$):

$$\begin{aligned}
\dot{\phi} &= \frac{1}{r}\bigg[\int_{\mathbb{R}} \Omega g(\Omega)\int_{\mathbb{T}} f_\Omega(\theta,t)\cos(\theta-\phi)d\theta d\Omega\\
&\qquad - \int_{\mathbb{R}} Krg(\Omega)\int_{\mathbb{T}} f_\Omega(\theta,t)\cos(\theta-\phi)\sin(\theta-\phi)d\theta d\Omega\bigg]\\
&= \int_{\mathbb{R}} \Omega g(\Omega)\frac{1}{r}\int_{\mathbb{T}} f_\Omega(\theta,t)\cos(\theta-\phi)d\theta d\Omega\\
&\quad - \frac{1}{2}\int_{\mathbb{R}} Kg(\Omega)\int_{\mathbb{T}} f_\Omega(\theta,t)\sin 2(\theta-\phi)d\theta d\Omega\\
&=: \mathcal{P}_1(t) + \mathcal{P}_2(t)
\end{aligned}$$

$$\begin{aligned}
|\mathcal{P}_1(t)| &\le \frac{1}{r}\int_{-M}^{M}\int_{\mathbb{T}} |\Omega||\cos(\theta-\phi)|f(\theta,\Omega,t)d\theta d\Omega\\
&\le \frac{1}{r}\int_{-M}^{M}\int_{\mathbb{T}} |\Omega| f(\theta,\Omega,t)d\theta d\Omega\\
&\le \frac{M}{r}\int_{\mathbb{R}}\int_{\mathbb{T}} f(\theta,\Omega,t)d\theta d\Omega = \frac{M}{r}
\end{aligned}$$

$$
\begin{aligned}
\mathcal{P}_2(t) &= -K\int_{\mathbb{R}}\int_{\mathbb{T}} f(\theta,\Omega,t)\sin(\theta-\phi)\cos(\theta-\phi)d\theta d\Omega \\
&= -K\int_{\mathbb{R}}\int_{\mathbb{T}} f(\theta,\Omega,t)\Big[\underbrace{\big(\sin(\theta-\phi)-1\big)\big(\cos(\theta-\phi)-1\big)}_{\geq 0} \\
&\qquad\qquad + \big(\cos(\theta-\phi)+\sin(\theta-\phi)-1\big)\Big]d\theta d\Omega \\
&\leq -K\int_{\mathbb{R}}\int_{\mathbb{T}} f(\theta,\Omega,t)\big(\cos(\theta-\phi)+\sin(\theta-\phi)-1\big)d\theta d\Omega \\
&= K(1-r)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{P}_2(t) &= -K\int_{\mathbb{R}}\int_{\mathbb{T}} f(\theta,\Omega,t)\sin(\theta-\phi)\cos(\theta-\phi)d\theta d\Omega \\
&= -K\int_{\mathbb{R}}\int_{\mathbb{T}} f(\theta,\Omega,t)\Big[\underbrace{\big(\sin(\theta-\phi)+1\big)\big(\cos(\theta-\phi)-1\big)}_{\leq 0} \\
&\qquad\qquad - \big(\cos(\theta-\phi)-\sin(\theta-\phi)-1\big)\Big]d\theta d\Omega \\
&\geq K\int_{\mathbb{R}}\int_{\mathbb{T}} f(\theta,\Omega,t)\big(\cos(\theta-\phi)-\sin(\theta-\phi)-1\big)d\theta d\Omega \\
&= -K(1-r)
\end{aligned}
$$

$$|\mathcal{P}_2(t)| \leq K(1-r)$$

Thererfore, we obtain

$$|\dot{\phi}| \leq \frac{M}{r} + K(1-r)$$

•(Estimates for r_0):

$$
\begin{aligned}
r(t) &= \int_{\mathbb{R}}\int_{\mathbb{T}} f(\theta,\Omega,t)\cos(\theta-\phi)d\theta d\Omega \\
&= \int_{\mathbb{R}}\int_{L_\gamma(t)} f(\theta,\Omega,t)\cos(\theta-\phi)d\theta d\Omega
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{\mathbb{R}}\int_{\mathbb{T}\setminus L_\gamma(t)} f(\theta,\Omega,t)\cos(\theta-\phi)d\theta d\Omega \\
&\geq \sin\gamma\int_{\mathbb{R}}\int_{L_\gamma(t)} f(\theta,\Omega,t)d\theta d\Omega - \int_{\mathbb{R}}\int_{\mathbb{T}\setminus L_\gamma(t)} f(\theta,\Omega,t)d\theta d\Omega \\
&= \sin\gamma\mathcal{F}_f\big(L_\gamma(t)\big) - \Big(1-\mathcal{F}_f\big(L_\gamma(t)\big)\Big) \\
&= (1+\sin\gamma)\mathcal{F}_f\big(L_\gamma(t)\big) - 1 \qquad (9.2.30)
\end{aligned}
$$

Thus,

$$
r_0 \geq (1+\sin\gamma)\mathcal{F}_f\big(L_\gamma(0)\big) - 1 > \varepsilon > 0.
$$

□

Proposition 9.2.1. *Define a functional* $\Gamma^1_\Omega(t) := \int_{L_\gamma(t)} f_\Omega(\theta,t)d\theta$*. Suppose the initial condition satisfies*

$$
\mathcal{F}_f\big(L_\gamma(0)\big) \geq \frac{2+\varepsilon+\cos\gamma}{(1+\cos\gamma)(1+\sin\gamma)}.
$$

Then, we attain

$$
\frac{d}{dt}\Gamma^1_\Omega(t) > 0
$$

for $K > \frac{M}{\varepsilon}(1+\frac{1}{\varepsilon})$*. Therefore, we obtain*

$$
\frac{d}{dt}\int_{\mathbb{R}}\int_{L_\gamma(t)} f(\theta,\Omega,t)d\theta d\Omega > 0.
$$

Proof.

$$
\begin{aligned}
&\frac{d}{dt}\Gamma^1_\Omega(t) \\
&= \dot\phi(t)\big(f_\Omega(\phi+\frac{\pi}{2}-\gamma,t) - f_\Omega(\phi-\frac{\pi}{2}+\gamma,t)\big) + \int_{L_\gamma(t)} \partial_t f_\Omega d\theta \\
&= \dot\phi(t)\big[B_{2,\Omega}(t)-B_{1,\Omega}(t)\big] - \int_{L_\gamma(t)} \partial_\theta\Big[f_\Omega(\theta,t)\big(\Omega-Kr\sin(\theta-\phi)\big)\Big]d\theta \\
&= \dot\phi(t)\big[B_{2,\Omega}(t)-B_{1,\Omega}(t)\big] \\
&\quad - B_{2,\Omega}(t)\big(\Omega-Kr\sin(\frac{\pi}{2}-\gamma)\big) + B_{1,\Omega}(t)\big(\Omega-Kr\sin(-\frac{\pi}{2}+\gamma)\big) \\
&= \dot\phi(t)\big[B_{2,\Omega}(t)-B_{1,\Omega}(t)\big]
\end{aligned}
$$

$$
\begin{aligned}
&- B_{2,\Omega}(t)\big(\Omega - Kr\cos\gamma\big) + B_{1,\Omega}(t)\big(\Omega + Kr\cos\gamma\big) \\
&= Kr\cos\gamma\big[B_{1,\Omega}(t) + B_{2,\Omega}(t)\big] + \big(\dot{\phi}(t) - \Omega\big)\big[B_{2,\Omega}(t) - B_{1,\Omega}(t)\big] \\
&\geq \big(Kr\cos\gamma - |\dot{\phi}(t) - \Omega|\big)\big|B_{1,\Omega}(t) + B_{2,\Omega}(t)\big| \\
&\geq \big(Kr\cos\gamma - M(1 + \frac{1}{r}) - K(1-r)\big)\big|B_{1,\Omega}(t) + B_{2,\Omega}(t)\big| \\
&= \Big(K\big(r(1+\cos\gamma) - 1\big) - M(1+\frac{1}{r})\Big)\big|B_{1,\Omega}(t) + B_{2,\Omega}(t)\big|.
\end{aligned}
$$

From (9.2.30), we have

$$
\begin{aligned}
\frac{d}{dt}\Gamma^1_\Omega(t) \geq |B_{1,\Omega}(t) + B_{2,\Omega}(t)| \\
\times \Bigg[K\Big\{\Big((1+\sin\gamma)\mathcal{F}_f\big(L_\gamma(t)\big) - 1\Big)(1+\cos\gamma) - 1\Big\} \\
- M\Big(1 + \frac{1}{(1+\sin\gamma)\mathcal{F}_f\big(L_\gamma(t)\big) - 1}\Big)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\Gamma^1_\Omega(0) \geq |B_{1,\Omega}(0) + B_{2,\Omega}(0)| \\
\times \Bigg[K\Big\{\Big((1+\sin\gamma)\mathcal{F}_f\big(L_\gamma(0)\big) - 1\Big)(1+\cos\gamma) - 1\Big\} \\
- M\Big(1 + \frac{1}{(1+\sin\gamma)\mathcal{F}_f\big(L_\gamma(0)\big) - 1}\Big)\Bigg].
\end{aligned}
$$

By the assumption (9.2.27),

$$
\Big((1+\sin\gamma)\mathcal{F}_f\big(L_\gamma(0)\big) - 1\Big)(1+\cos\gamma) - 1 > \varepsilon > 0.
$$

By (9.2.28), we attain

$$
\frac{1}{(1+\sin\gamma)\mathcal{F}_f\big(L_\gamma(0)\big) - 1} \leq \frac{1}{\varepsilon}.
$$

Therefore, for $K > \frac{M}{\varepsilon}(1 + \frac{1}{\varepsilon})$, we obtain

$$
\frac{d}{dt}\Gamma^1_\Omega(0) \geq |B_{1,\Omega}(0) + B_{2,\Omega}(0)|\Big(K\varepsilon - M\big(1 + \frac{1}{\varepsilon}\big)\Big) > 0.
$$

Thus, there exists time $t_* > 0$ such that

$$\frac{d}{dt}\Gamma^1_\Omega(t) > 0$$

for $0 \le t < t_*$. Let $\mathcal{T} := \{t_* \in [0,\infty) : \frac{d}{dt}\Gamma^1_\Omega(t) > 0 \quad \text{for all} \quad t \in (0,t_*)\}$ and set $T := \sup\mathcal{T}$. We claim that $T = \infty$. Suppose $T < \infty$. By (9.2.30), we have

$$\begin{aligned} r(T) &\ge (1+\sin\gamma)\mathcal{F}_f\big(L_\gamma(T)\big) - 1 \\ &\ge (1+\sin\gamma)\mathcal{F}_f\big(L_\gamma(0)\big) - 1 \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt}\Gamma^1_\Omega(T) \ge{}& \Big(K\big(r(1+\cos\gamma)-1\big) - M(1+\frac{1}{r})\Big)\big|B_{1,\Omega}(T)+B_{2,\Omega}(T)\big| \\ ={}& |B_{1,\Omega}(T)+B_{2,\Omega}(T)| \\ &\times\Bigg[K\Big\{\Big((1+\sin\gamma)\mathcal{F}_f\big(L_\gamma(T)\big)-1\Big)(1+\cos\gamma)-1\Big\} \\ &\quad - M\Big(1+\frac{1}{(1+\sin\gamma)\mathcal{F}_f\big(L_\gamma(T)\big)-1}\Big)\Bigg] \\ \ge{}& |B_{1,\Omega}(T)+B_{2,\Omega}(T)| \\ &\times\Bigg[K\Big\{\Big((1+\sin\gamma)\mathcal{F}_f\big(L_\gamma(0)\big)-1\Big)(1+\cos\gamma)-1\Big\} \\ &\quad - M\Big(1+\frac{1}{(1+\sin\gamma)\mathcal{F}_f\big(L_\gamma(0)\big)-1}\Big)\Bigg] \\ ={}& |B_{1,\Omega}(T)+B_{2,\Omega}(T)|\bigg(K\varepsilon - M\big(1+\frac{1}{\varepsilon}\big)\bigg) \\ >{}& 0 \quad \text{for} \quad K > \frac{M}{\varepsilon}(1+\frac{1}{\varepsilon}). \end{aligned}$$

By the continuity, there exists time $\tilde{t}$ such that $\frac{d}{dt}\Gamma^1_\Omega(t) > 0$ for $T < t < \tilde{t}$. This contradicts to the definition of T. Thus, $T = \infty$ Hence,

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}}\int_{L_\gamma(t)} f(\theta,\Omega,t)\,d\theta d\Omega \\ &= \int_{\mathbb{R}} g(\Omega)\left[\frac{d}{dt}\int_{L_\gamma(t)} f_\Omega(\theta,t)\,d\theta\right]d\Omega \end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} g(\Omega)\left[\frac{d}{dt}\Gamma^1_{\Omega}(t)\right]d\Omega \\
&> 0.
\end{aligned}
$$

□

Proposition 9.2.2. *We set a functional* $\Gamma^2_{\bar{\Omega}}(t) := \int_{L_\gamma(t)} |f_{\bar{\Omega}}(\theta,t)|^2 d\theta$. *Suppose the initial condition satisfies*

$$
\mathcal{F}_f\big(L_\gamma(0)\big) \geq \frac{2+\varepsilon+\cos\gamma}{(1+\cos\gamma)(1+\sin\gamma)}.
$$

Then, we have

$$
\Gamma^2_{\bar{\Omega}}(t) \geq \Gamma^2_{\bar{\Omega}}(0)e^{K\varepsilon\sin\gamma t}
$$

for $K > \frac{M}{\varepsilon}\big(1+\frac{1}{\varepsilon}\big)$

Proof. We take time derivative to attain

$$
\begin{aligned}
\frac{d}{dt}\Gamma^2_{\bar{\Omega}}(t) &= \dot{\phi}(t)\big(B_{2,\bar{\Omega}}(t)\big)^2 - \dot{\phi}(t)\big(B_{1,\bar{\Omega}}(t)\big)^2 + 2\int_{L_\gamma(t)} f_{\bar{\Omega}}\partial_t f_{\bar{\Omega}} d\theta \\
&=: \dot{\phi}(t)\Big[\big(B_{2,\bar{\Omega}}(t)\big)^2 - \big(B_{1,\bar{\Omega}}(t)\big)^2\Big] + \mathcal{L}_{\bar{\Omega}}(t)
\end{aligned}
$$

By (9.2.24), we have

$$
\begin{aligned}
\mathcal{L}_{\bar{\Omega}}(t) &= -2\int_{L_\gamma(t)} f_{\bar{\Omega}}\partial_\theta\Big[f_{\bar{\Omega}}\big(\bar{\Omega} - Kr\sin(\theta-\phi)\big)\Big]d\theta \\
&= -2\int_{L_\gamma(t)} (f_{\bar{\Omega}}\partial_\theta f_{\bar{\Omega}})\big(\bar{\Omega} - Kr\sin(\theta-\phi)\big) - (f^2_{\bar{\Omega}})Kr\cos(\theta-\phi)d\theta \\
&= -\int_{L_\gamma(t)} (\partial_\theta\big(f_{\bar{\Omega}})^2\big)\big(\bar{\Omega} - Kr\sin(\theta-\phi)\big)d\theta \\
&\quad + \int_{L_\gamma(t)} 2(f^2_{\bar{\Omega}})Kr\cos(\theta-\phi)d\theta \\
&=: \mathcal{L}_1(t) + \mathcal{L}_2(t),
\end{aligned}
$$

$$\begin{aligned}
&\mathcal{L}_1(t)\\
&= -\Big[\big(B_{2,\bar\Omega}(t)\big)^2\big(\bar\Omega - Kr\sin(\frac{\pi}{2}-\gamma)\big) - \big(B_{1,\bar\Omega}(t)\big)^2\big(\bar\Omega - Kr\sin(-\frac{\pi}{2}+\gamma)\big)\Big]\\
&\quad - Kr\int_{L_\gamma(t)}\big(f_{\bar\Omega}(\theta,t)\big)^2\cos(\theta-\phi)d\theta\\
&=: \mathcal{L}_{11}(t) + \mathcal{L}_{12}(t)
\end{aligned}$$

where we use integration by parts. We have the following relations for $\mathcal{L}_1(t)$ and $\mathcal{L}_2(t)$

$$\begin{aligned}
&\mathcal{L}_{11}(t) = -\bar\Omega\Big[\big(B_{2,\bar\Omega}(t)\big)^2 - \big(B_{1,\bar\Omega}(t)\big)^2\Big] + Kr\cos\gamma\Big[\big(B_{2,\bar\Omega}(t)\big)^2 + \big(B_{1,\bar\Omega}(t)\big)^2\Big]\\
&\mathcal{L}_{12}(t) + \mathcal{L}_2(t) \geq Kr\Gamma^2_{\bar\Omega}(t)\sin\gamma.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\frac{d}{dt}\Gamma^2_{\bar\Omega}(t) &\geq (\dot\phi(t)-\bar\Omega)\Big[\big(B_{2,\bar\Omega}(t)\big)^2 - \big(B_{1,\bar\Omega}(t)\big)^2\Big]\\
&\quad + Kr\cos\gamma\Big[\big(B_{2,\bar\Omega}(t)\big)^2 + \big(B_{1,\bar\Omega}(t)\big)^2\Big] + Kr\Gamma^2_{\bar\Omega}(t)\sin\gamma\\
&\geq \big(Kr\cos\gamma - |\dot\phi(t)-\bar\Omega|\big)\Big[\big(B_{2,\bar\Omega}(t)\big)^2 + \big(B_{1,\bar\Omega}(t)\big)^2\Big] + Kr\Gamma^2_{\bar\Omega}(t)\sin\gamma\\
&\geq \Big(Kr\cos\gamma - M(1+\frac{1}{r}) - K(1-r)\Big)\Big[\big(B_{2,\bar\Omega}(t)\big)^2 + \big(B_{1,\bar\Omega}(t)\big)^2\Big]\\
&\quad + Kr\Gamma^2_{\bar\Omega}(t)\sin\gamma\\
&\geq \Big[\big(B_{2,\bar\Omega}(t)\big)^2 + \big(B_{1,\bar\Omega}(t)\big)^2\Big]\\
&\qquad \times \Big(K\big(r(1+\cos\gamma)-1\big) - M(1+\frac{1}{r})\Big)\\
&\quad + Kr\Gamma^2_{\bar\Omega}(t)\sin\gamma\\
&\geq \Big[\big(B_{2,\bar\Omega}(t)\big)^2 + \big(B_{1,\bar\Omega}(t)\big)^2\Big]\\
&\qquad \times \bigg\{K\Big(\big((1+\sin\gamma)\mathcal{F}_f\big(L_\gamma(t)\big)-1\big)(1+\cos\gamma)-1\Big)\\
&\qquad\qquad - M\bigg(1+\frac{1}{(1+\sin\gamma)\mathcal{F}_f\big(L_\gamma(t)\big)-1}\bigg)\bigg\}\\
&\quad + K\Big((1+\sin\gamma)\mathcal{F}_f\big(L_\gamma(t)\big)-1\Big)\Gamma^2_{\bar\Omega}(t)\sin\gamma
\end{aligned}$$

$$
\begin{aligned}
&\geq \Big[\big(B_{2,\bar{\Omega}}(t)\big)^2 + \big(B_{1,\bar{\Omega}}(t)\big)^2\Big] \\
&\qquad \times \bigg\{K\Big(\big((1+\sin\gamma)\mathcal{F}_f\big(L_\gamma(0)\big) - 1\big)(1+\cos\gamma) - 1\Big) \\
&\qquad\qquad - M\bigg(1 + \frac{1}{(1+\sin\gamma)\mathcal{F}_f\big(L_\gamma(0)\big) - 1}\bigg)\bigg\} \\
&\quad + K\Big((1+\sin\gamma)\mathcal{F}_f\big(L_\gamma(0)\big) - 1\Big)\Gamma^2_{\bar{\Omega}}(t)\sin\gamma \\
&\geq \Big[\big(B_{2,\bar{\Omega}}(t)\big)^2 + \big(B_{1,\bar{\Omega}}(t)\big)^2\Big]\Big\{K\varepsilon - M\big(1 + \frac{1}{\varepsilon}\big)\Big\} + K\varepsilon\Gamma^2_{\bar{\Omega}}(t)\sin\gamma \\
&\geq K\varepsilon\Gamma^2_{\bar{\Omega}}(t)\sin\gamma.
\end{aligned}
$$

Gronwall's inequality yields the desired results. $\square$

Chapter 10

Conclusion and future works

This chapter is devoted for closing remarks to the Kuramoto model and proposing open problems. The Kuramoto model is a prototype model to describe the synchronization phenomena. From Chapter 3 to Chapter 7, we studied the emergence of synchronization for the classical Kuramoto model in various circumstances, such as relaxed initial constraints, network structures, frustrations, external heterogeneous forcings, inertia, and adaptive couplings, etc. We provided the sufficient conditions for initial configurations to achieve synchronization. We use Lyapunov functional approach extensively for the analysis and these estimates are studied under technical constraints. However, we can observe numerically that the synchronous behavior can be achieved with further relaxed conditions. In [38], the authors proved the complete synchronization with generic initial configurations under the large coupling strength by using the gradient flow structure of the Kuramoto system. We may improve the aforementioned results with some alternative methods.

We also studied the dynamics for the kinetic Kuramoto model which represents the mean field limit of the Kuramoto oscillators in Chapters 8 and 9. We showed the existence of BV weak solution by using front tracking algorithm and provide the exponential growth of the approximate solution. Under absolutely continuous framework, we show the concentration of oscillators by focusing on the dynamics of kinetic Kuramoto order parameters. As in the particle model, we expect to extend the current results to relaxed or generic initial condition by studying the inherent structure of the Kuramoto-

Sakaguchi equation.

We next propose some related open problems in the following.

1. We can generalize the Kuramoto model by replacing sine function to general 2π-periodic functions:

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N}\Gamma(\theta_j - \theta_i),$$

   where Γ is 2π periodic. Especially, if Γ is π-periodic, for example, $\Gamma(\theta) = \sin 2\theta$, then the dynamics has attraction and repulsion forces. We may consider the dynamics of the coupled system with various geometric conditions on Γ.

2. If we expand the sine term in R.H.S,

$$\begin{aligned}\dot{\theta}_i &= \Omega_i + \frac{K}{N}\cos\theta_i\sum_{j=1}^{N}\sin\theta_j - \frac{K}{N}\sin\theta_i\sum_{j=1}^{N}\cos\theta_j \\ &=: \Omega_i + \frac{K}{N}S_1(\theta_i)\sum_{j=1}^{N}I_1(\theta_j) - \frac{K}{N}S_2(\theta_i)\sum_{j=1}^{N}I_2(\theta_j),\end{aligned}$$

   then, the Kuramoto model can be expressed by the Winfree type model with two interaction terms. In general, the Winfree model shows more diverse dynamics compared to the Kuramoto model. We can apply variations such as adaptive couplings and heterogeneous external forcings into the Winfree model.

3. As we study for the identical Kuramoto-Sakaguchi equation, we may apply the front tracking method for the non-identical case. By discretizing the distribution of natural frequency $g(\Omega)$, we can define an approximate solution for the non-identical Kuramoto-Sakaguchi equation. Such numerical approaches have merits that we can observe the structures of the solutions.

4. By adopting the inertia to the kinetic Kuramoto model, we can induce the following kinetic equation for a one-particle distribution function $f = f(\theta, \omega, \Omega, t)$:

$$\begin{aligned}
&\partial_t f + \partial_\theta(\omega f) + \partial_\omega(V[f]f) = 0,\\
&V[f](\theta, \omega, \Omega, t)\\
&= \frac{1}{m}\Big\{ -\omega + \Omega - K \iiint \sin(\theta - \theta_*) f(\theta_*, \omega_*, \Omega_*, t) d\theta_* d\omega_* d\Omega_* \Big\}
\end{aligned} \tag{10.0.1}$$

where $\omega = \frac{d\theta}{dt}$. The equation (10.0.1) reduces to the kinetic Kuramoto model without inertia (2.3.4) by vanishing inertia $m \to 0$ [9]. The asymptotic dynamics for the kinetic Kuramoto model with inertia is also an open problem.

Appendix A

Positivity of t_e and ε_θ in Lemma 4.4.6

In this appendix, we prove the positivity of t_e and ε_θ defined in (4.4.35) as a solution of the system (4.4.34).

Recall that

$$t_e = \frac{2\beta_\delta - D_2^\infty + \alpha}{2C_1 + C_2}, \quad \varepsilon_\theta = \frac{C_1(D_2^\infty - \alpha - \pi) + C_2(\beta_\delta - \frac{\pi}{2})}{2C_1 + C_2},$$

where C_1 and C_2 are given by the following relations:

$$\begin{aligned} C_1 &:= K[1 - r_0(1 - 2\sin\alpha)] + \Big(1 + \frac{1}{r_0}\Big) \max_{1\le j\le N} |\Omega_j|, \\ C_2 &:= KA(D_2^\infty - \alpha)r_0 - D(\Omega), \quad A := \cos\alpha \frac{\sin\beta_\delta}{\beta_\delta} - \sin\alpha. \end{aligned}$$

• Case A (Positivity of the term $2C_1 + C_2$): It follows from (6.1.1) that we have

$$\begin{aligned} &2C_1 + C_2 \\ &= 2K[1 - r_0(1 - 2\sin\alpha)] + 2\Big(1 + \frac{1}{r_0}\Big)(\max_{1\le j\le N} |\Omega_j|) \\ &+ KA(D_2^\infty - \alpha)r_0 - D(\Omega) \end{aligned}$$

$$
\begin{aligned}
&= K\Big[2(1-r_0) + 4r_0\sin\alpha + A(D_2^\infty - \alpha)r_0\Big] \\
&+ 2\Big(1 + \frac{1}{r_0}\Big)\max_{1\le j\le N}|\Omega_j| - D(\Omega) \\
&\ge 0.
\end{aligned}
\tag{A.0.1}
$$

Here we used the simple relations:

$$
0 \le r_0 \le 1, \quad D(\Omega) \le 2\max_{1\le j\le N}|\Omega_j|.
$$

Thus, the denominators of t_e and ε_θ are positive.

• Case B (Positivity of t_e): Because $\delta \in (0, \frac{1}{2})$, we have

$$
2\beta_\delta = 2(1-\delta)\pi > \pi, \quad D_2^\infty - \alpha < \pi.
$$

Thus, we have

$$
2\beta_\delta - D_2^\infty + \alpha > 0, \quad \text{i.e.,} \quad t_e > 0.
$$

• Case C (Positivity of ε_θ): By direct calculation, we have

$$
\begin{aligned}
&C_1(D_2^\infty - \alpha - \pi) + C_2(\beta_\delta - \frac{\pi}{2}) \\
&= (D_2^\infty - \alpha - \pi)K(1 - r_0(1 - 2\sin\alpha)) + (D_2^\infty - \alpha - \pi)\Big(1 + \frac{1}{r_0}\Big)\max_{1\le j\le N}|\Omega_j| \\
&+ (\beta_\delta - \frac{\pi}{2})\Big[KA(D_2^\infty - \alpha)r_0 - D(\Omega)\Big] \\
&= K\underbrace{\Big[(D_2^\infty - \alpha - \pi)(1 - r_0(1 - 2\sin\alpha)) + Ar_0(\beta_\delta - \frac{\pi}{2})(D_2^\infty - \alpha)\Big]}_{=:\mathcal{M}_1} \\
&- \underbrace{\Big((\pi + \alpha - D_2^\infty)(1 + \frac{1}{r_0})\max_j|\Omega_j| + (\beta_\delta - \frac{\pi}{2})D(\Omega)\Big)}_{=:\mathcal{M}_2}.
\end{aligned}
\tag{A.0.2}
$$

Then, because $\pi > D_2^\infty - \alpha$ and $\beta_\delta > \frac{\pi}{2}$, we have

$$
\mathcal{M}_2 > 0.
$$

We claim:

$$
\mathcal{M}_1 > 0. \tag{A.0.3}
$$

The proof of claim (A.0.3):

$$\mathcal{M}_1 = r_0\Big[(\pi+\alpha-D_2^\infty)(1-2\sin\alpha)+A(\beta_\delta-\frac{\pi}{2})(D_2^\infty-\alpha)\Big]-(\pi+\alpha-D_2^\infty)$$

We define

$$\tilde{r} := \frac{(\pi+\alpha-D_2^\infty)}{(\pi+\alpha-D_2^\infty)(1-2\sin\alpha)+A(\beta_\delta-\frac{\pi}{2})(D_2^\infty-\alpha)} \tag{A.0.4}$$

and we will show that $\tilde{r} < r_*$ for large K due to the smallness of α so that the condition $r_0 > r_*$ yields $\mathcal{M}_1 > 0$. For simplicity of notation, let $\|\Omega\| := \max_{1\le j\le N}|\Omega_j|$. By the definition of D_2^∞, we have

$$\frac{1-\cos\alpha}{\frac{\pi}{2}-\alpha}(\pi-D_2^\infty-\alpha) < \frac{\|\Omega\|}{K} = \sin D_2^\infty - \sin\alpha < \cos\alpha(\pi-D_2^\infty-\alpha). \tag{A.0.5}$$

From (A.0.5), we have upper and lower bounds for D_2^∞

$$(\pi-\alpha)-\frac{(\frac{\pi}{2}-\alpha)\|\Omega\|}{(1-\cos\alpha)K} < D_2^\infty < (\pi-\alpha)-\frac{\|\Omega\|}{K\cos\alpha}.$$

Thus,

$$2\alpha+\frac{\|\Omega\|}{K\cos\alpha} < \pi+\alpha-D_2^\infty < 2\alpha+\frac{(\frac{\pi}{2}-\alpha)\|\Omega\|}{(1-\cos\alpha)K}. \tag{A.0.6}$$

By applying (A.0.6) to (A.0.4), we have the following inequality

$$\begin{aligned}\tilde{r} &< \frac{2\alpha+\frac{(\frac{\pi}{2}-\alpha)\|\Omega\|}{(1-\cos\alpha)K}}{(2\alpha+\frac{\|\Omega\|}{K\cos\alpha})(1-2\sin\alpha)+(\beta_\delta-\frac{\pi}{2})A(\pi-2\alpha-\frac{(\frac{\pi}{2}-\alpha)\|\Omega\|}{(1-\cos\alpha)K})}\\ &= \frac{2\alpha K+\|\Omega\|\frac{\frac{\pi}{2}-\alpha}{1-\cos\alpha}}{K\big[2\alpha(1-2\sin\alpha)+(\beta_\delta-\frac{\pi}{2})A(\pi-2\alpha)\big]+\|\Omega\|\big[\frac{1-2\sin\alpha}{\cos\alpha}-(\beta_\delta-\frac{\pi}{2})A\frac{\frac{\pi}{2}-\alpha}{1-\cos\alpha}\big]}\end{aligned}$$

We recall that $r_* = \frac{1}{\sqrt{\bar{R}}\sin\beta_\delta}\big(\frac{\|\Omega\|}{K}+\alpha\big)$. There exists a $\bar{K}_1$ that satisfies $r_* > \tilde{r}$ for $K > \bar{K}_1$ as long as the following inequality holds

$$\frac{2\alpha}{2\alpha(1-2\sin\alpha)+(\beta_\delta-\frac{\pi}{2})A(\pi-2\alpha)} < \frac{\alpha}{\sqrt{\bar{R}}\sin\beta_\delta}. \tag{A.0.7}$$

Because
$$\bar{R} < (1 - \frac{\pi}{2\beta_\delta})^2 < \frac{\pi^2}{8}(1 - \frac{\pi}{2\beta_\delta})^2,$$
we have
$$2\sqrt{2\bar{R}} < \pi(1 - \frac{\pi}{2\beta_\delta}),$$
or, equivalently,
$$\frac{1}{\beta_\delta} - \frac{2\sqrt{2\bar{R}}}{(\beta_\delta - \frac{\pi}{2})\pi} > 0.$$

We assume that α is small enough to satisfy
$$0 < \alpha < \min\{\frac{\pi}{4}, \quad \frac{\pi}{4}\sin\beta_\delta\Big[\frac{1}{\beta_\delta} - \frac{2\sqrt{2\bar{R}}}{(\beta_\delta - \frac{\pi}{2})\pi}\Big]\}.$$

Then, we have
$$\tan\alpha < \frac{4}{\pi}\alpha < \sin\beta_\delta\Big[\frac{1}{\beta_\delta} - \frac{2\sqrt{2\bar{R}}}{(\beta_\delta - \frac{\pi}{2})\pi}\Big],$$

or, in other words,
$$2\sqrt{2\bar{R}}\sin\beta_\delta < (\beta_\delta - \frac{\pi}{2})\pi(\frac{\sin\beta_\delta}{\beta_\delta} - \tan\alpha).$$

By multiplying by $\frac{1}{\sqrt{2}} < \cos\alpha < 1$ on both sides, we obtain
$$\begin{aligned} &2\sqrt{\bar{R}}\sin\beta_\delta - (\beta_\delta - \frac{\pi}{2})\pi(\cos\beta_\delta\frac{\sin\beta_\delta}{\beta_\delta} - \sin\alpha) \\ &= 2\sqrt{\bar{R}}\sin\beta_\delta - (\beta_\delta - \frac{\pi}{2})\pi A < 0 \end{aligned} \tag{A.0.8}$$

To show (A.0.7), it suffices to show that
$$\frac{2\alpha}{2\alpha(1-2\alpha) + (\beta_\delta - \frac{\pi}{2})A(\pi - 2\alpha)} < \frac{\alpha}{\sqrt{\bar{R}}\sin\beta_\delta},$$

or, equivalently,
$$4\alpha^2 - 2\big(1 - (\beta_\delta - \frac{\pi}{2})A\big)\alpha + 2\sqrt{\bar{R}}\sin\beta_\delta - (\beta_\delta - \frac{\pi}{2})\pi A < 0. \tag{A.0.9}$$

Consider the second order equation

$$4x^2 - 2\big(1 - (\beta_\delta - \frac{\pi}{2})A\big)x + 2\sqrt{\bar{R}}\sin\beta_\delta - (\beta_\delta - \frac{\pi}{2})\pi A = 0, \qquad \text{(A.0.10)}$$

and let α_1 and α_2 be the two solutions of (A.0.10). From (A.0.8), we have $\alpha_1\alpha_2 < 0$; Thus (A.0.10) has two real roots. We have

$$\begin{aligned}
\frac{\alpha_1 + \alpha_2}{2} &= \frac{1}{4}\big(1 - (\beta_\delta - \frac{\pi}{2})A\big) \\
&= \frac{1}{4}\big(1 - (\beta_\delta - \frac{\pi}{2})(\cos\alpha\frac{\sin\beta_\delta}{\beta_\delta} - \sin\alpha)\big) \\
&\geq \frac{1}{4}\big(1 - (\beta_\delta - \frac{\pi}{2})\cos\alpha\frac{\sin\beta_\delta}{\beta_\delta}\big) \\
&\geq \frac{1}{4}\big(1 - (1 - \frac{\pi}{2\beta_\delta})\big) = \frac{\pi}{8\beta_\delta} > \frac{1}{8}.
\end{aligned}$$

Because (A.0.10) is symmetric with respect to $\frac{\alpha_1+\alpha_2}{2}$, (A.0.9) holds for $\alpha < \frac{1}{4}$. Therefore, if we choose α to satisfy

$$0 < \alpha < \min\Big\{\frac{1}{4}, \quad \frac{\pi}{4}\sin\beta_\delta\Big[\frac{1}{\beta_\delta} - \frac{2\sqrt{2\bar{R}}}{(\beta_\delta - \frac{\pi}{2})\pi}\Big]\Big\},$$

we can conclude that (A.0.9) is true, i.e., $r_* > \tilde{r}$ for $K > \bar{K}_1$. Hence, the initial constraint $r_0 > r_*$ guarantees that $r_0 > \tilde{r}$, and, eventually, $\mathcal{M}_1 > 0$. We now set K_1:

$$\tilde{K}_1 := \frac{\mathcal{M}_2}{\mathcal{M}_1} > 0.$$

Then, it follows from (A.0.2) that if $K > K_1 := \max\{\tilde{K}_1, \bar{K}_1\}$, we have

$$C_1(D_2^\infty - \alpha - \pi) + C_2(\beta_\delta - \frac{\pi}{2}) > 0. \qquad \text{(A.0.11)}$$

Finally, it follows from (A.0.1) and (A.0.11) that we have

$$\varepsilon_\theta > 0.$$

Appendix B

Proof of Lemma 5.2.3

In this appendix, we provide the proof of Lemma 5.2.3 using elementary row operations and the Laplace expansion.

$$
\begin{aligned}
\det M_N =& \det \begin{pmatrix} p_1 - \frac{N-1}{N}K & \frac{K}{N} & \frac{K}{N} & \dots & \frac{K}{N} \\ \frac{K}{N} & p_2 - \frac{N-1}{N}K & \frac{K}{N} & \dots & \frac{K}{N} \\ \frac{K}{N} & \frac{K}{N} & p_3 - \frac{N-1}{N}K & \dots & \frac{K}{N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{K}{N} & \frac{K}{N} & \frac{K}{N} & \dots & p_N - \frac{N-1}{N}K \end{pmatrix} \\
=& \det \begin{pmatrix} p_1 - \frac{N-1}{N}K & \frac{K}{N} & \frac{K}{N} & \dots & \frac{K}{N} \\ -(p_1 - K) & p_2 - K & 0 & \dots & 0 \\ -(p_1 - K) & 0 & p_3 - K & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -(p_1 - K) & 0 & 0 & \dots & p_N - K \end{pmatrix} \\
=& (p_1 - \frac{N-1}{N}K) \det \begin{pmatrix} p_2 - K & 0 & \dots & 0 \\ 0 & p_3 - K & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & p_N - K \end{pmatrix}
\end{aligned}
$$

$$
-(-1)(p_1 - K)\det\begin{pmatrix} \frac{K}{N} & \frac{K}{N} & \frac{K}{N} & \dots & \frac{K}{N} \\ 0 & p_3 - K & 0 & \dots & 0 \\ 0 & 0 & p_4 - K & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & p_N - K \end{pmatrix}
$$

$$
+(-1)(p_1 - K)\det\begin{pmatrix} \frac{K}{N} & \frac{K}{N} & \frac{K}{N} & \dots & \frac{K}{N} \\ p_2 - K & 0 & 0 & \dots & 0 \\ 0 & 0 & p_4 - K & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & p_N - K \end{pmatrix} + \cdots
$$

$$
+(-1)^{N-1}(-1)(p_1 - K)\det\begin{pmatrix} \frac{K}{N} & \frac{K}{N} & \dots & \frac{K}{N} & \frac{K}{N} \\ p_2 - K & 0 & \dots & 0 & 0 \\ 0 & p_3 - K & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & p_{N-1} - K & 0 \end{pmatrix}
$$

$$
=: \mathcal{M}_1 + \cdots + \mathcal{M}_N.
$$

We next estimate the $\mathcal{M}_i$ separately.

• (Estimate of $\mathcal{M}_1$):

$$
\begin{aligned}
\mathcal{M}_1 &= \Big(p_1 - \frac{N-1}{N}K\Big)\prod_{i=2}^{N}(p_i - K) \\
&= (-1)^N \frac{N-1}{N} K^N \\
&+ \{(-1)^{N-1}p_1 + (-1)^{N-1}\frac{N-1}{N}(\sum_{i=2}^{N} p_i)\}K^{N-1} + \mathcal{O}(K^{N-2}).
\end{aligned}
$$

• (Estimate of $\mathcal{M}_2$):

$$
\begin{aligned}
\mathcal{M}_2 &= (p_1 - K)\frac{K}{N}(p_3 - K)(p_4 - K)\cdots(p_N - K) \\
&= \frac{1}{N}(-1)^{N-1}K^N + (-1)^{N-2}(p_1 + p_3 + p_4 + \cdots + p_N)K^{N-1} + \mathcal{O}(K^{N-2}).
\end{aligned}
$$

• (Estimate of $\mathcal{M}_3$):

$$\begin{aligned}
\mathcal{M}_3 &= (p_1 - K)\frac{K}{N}(p_2 - K)(p_4 - K)\cdots(p_N - K) \\
&= \frac{1}{N}(-1)^{N-1}K^N + (-1)^{N-2}(p_1 + p_2 + p_4 + \cdots + p_N)K^{N-1} + \mathcal{O}(K^{N-2}). \\
&\vdots
\end{aligned}$$

• (Estimate of $\mathcal{M}_N$):

$$\begin{aligned}
\mathcal{M}_N &= (p_1 - K)\frac{K}{N}(p_2 - K)(p_3 - K)\cdots(p_{N-1} - K) \\
&= \frac{1}{N}(-1)^{N-1}K^N \\
&+ (-1)^{N-2}(p_1 + p_2 + p_3 + \cdots + p_{N-1})K^{N-1} + \mathcal{O}(K^{N-2}).
\end{aligned}$$

Hence, we have

$$\det M_N = \sum_{i=1}^{N} \mathcal{M}_i = \frac{(-1)^{N-1}}{N}(p_1 + \cdots + p_N)K^{N-1} + \mathcal{O}(K^{N-2}).$$

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국문초록

이 논문에서 우리는 동기화 현상을 기술하는 쿠라모토 모델에 관하여 연구한다. 쿠라모토 모델에서는 진동자들의 동역학이 내재적인 상수 역동성과 진동자간의 결합으로 주어진다. 우리는 네트워크 구조, 결합 노이즈, 이질적인 내재 역동성, 관성 효과 등의 다양한 상황에서 쿠라모토 모델의 동기화를 얻기위한 충분조건에 대해서 알아본다. 또한, 쿠라모토 모델의 평균장 극한에 대한 거시적 표현인 쿠라모도-사카구치 운동방정식의 동역학에 대해서도 연구한다.

주요어휘: 쿠라모토 모델, 운동 방정식, 쿠라모토-사카구치 방정식, 동기화, 동역학계
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